Vol 14 No 7, July 2005
1009-1963/2005/14(07)/1342-05

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

Generating chaos in the sense of Devaney via decentralized feedback control from a stable Takagi–Sugeno fuzzy system*

Lu Jun-Guo(卢俊国)†

Department of Automation, Shanghai Jiaotong University, Shanghai 200030, *China*

(Received 25 November 2004; revised manuscript received 24 January 2005)

The problem of making a stable Takagi–Sugeno (TS) fuzzy system chaotic in the sense of Devaney by using decentralized state feedback control is studied. The proposed decentralized feedback controller is a decentralized linear feedback controller composed of the overflow nonlinear function of the 2's complement arithmetic. The overflow nonlinearity of the 2's complement arithmetic results in a rather complex dynamics. We apply the Shi–Chen theorem to mathematically prove that the controlled system is indeed chaotic in the sense of Devaney. In particular, an explicit formula for the computation of chaotification parameters is also obtained. A numerical example is used to visualize and illustrate the theoretical results.

Keywords: chaotification, Devaney chaos, TS fuzzy system, two complement arithmetic
PACC: 0545

1. Introduction

In addition to the rapid development of chaos suppression and chaos synchronization in the last decade,[1,2] chaotification (also called "anticontrol of chaos") by means of making an originally non-chaotic dynamical system chaotic, or enhancing and retaining existing chaos, has attracted special attention from the engineering and physics communities lately due to the great potential of chaos in nontraditional applications.[2,3] Up to now, most of the chaotification algorithms[3−15] use the definition of chaos introduced by Li and Yorke[16] as their design criteria. There is yet another mathematical definition of chaos introduced by Devaney.[17] In this paper, we use the mathematical definition of chaos introduced by Devaney[17] as the design criterion.

Recently, Li *et al*[14] proposed a method for chaotifying discrete-time Takagi–Sugeno (TS) fuzzy systems and applied the Marotto theorem[18] to prove that their approach could generate chaos in the sense of Li and Yorke[16] with the control $u_k = [\sigma \sin(\pi\beta\, x_k(1)/\sigma), \cdots, \sigma \sin(\pi\beta\, x_k(n)/\sigma)]^{\mathrm{T}}$ if $\beta > \bar{\beta}$ for a sufficiently large $\bar{\beta} > 0$ and $\sigma > 0$, but they did not tell us how to obtain $\bar{\beta}$. Moreover, Refs.[19–21] have pointed out that the Marotto theorem[18] has a problem since Marotto had used the following property $A' \Rightarrow B'$ in his theorem:

A' : All eigenvalues of the real matrix A are larger than 1 in absolute value;

B' : A satisfies $\|Ax\|_2 \geq \mu\|x\|_2$ for some $\mu > 1$, for all $x \in R^n$.

Later, in Ref.[12], we used the decentralized feedback controller that is a simple sawtooth function of the system states to chaotifying discrete-time TS fuzzy systems and applied the corrected Marotto theorem to mathematically prove that the controlled system is chaotic in the sense of Li and Yorke.[16] In particular, an explicit formula for the computation of chaotification parameters was also obtained.

Chua and Lin have shown in Ref.[22] that the second-order digital filter, when implemented using a 2's complement arithmetic, can exhibit chaotic behaviour for certain region in the parameter space. The question addressed in this paper is: can we design a control with the overflow nonlinearity of the above 2's complement arithmetic to make a stable TS fuzzy sys-

*Project supported by the National Postdoctoral Science Foundation of China and the National Natural Science Foundation of China (Grant No 60404005).

†Corresponding author. E-mail: jglu@sjtu.edu.cn

tem chaotic in the sense of Devaney? If so, is there a practical and systematic way to achieve it? In this paper we attempt to provide positive answers to the two questions. More precisely, for an arbitrarily given stable TS fuzzy system, we propose a decentralized feedback chaotification method, in which the designed decentralized feedback controller is a decentralized linear feedback controller composed of the overflow nonlinear function of the 2's complement arithmetic. Based on the Shi–Chen theorem,[23] we mathematically prove that the controlled system is indeed chaotic in the sense of Devaney.[17] In particular, an explicit formula for the computation of chaotification parameters is also obtained. A numerical example is used to visualize and illustrate the theoretical results.

Throughout this paper, all vector inequalities such as $x \geq 0$ are component-wise inequalities, $b = [1, \ 1, \ \ldots, \ 1]^{\mathrm{T}}$. For a vector $x \in R^n$, its infinity-norm is defined by $\|x\|_\infty = \max\limits_{1\leq i\leq n} |x_i|$, and its corresponding induced norm is defined respectively by $\|A\|_\infty = \max\limits_{1\leq i\leq n} \left\{ \sum\limits_{j=1}^{n} |a_{ij}| \right\}$.

2. Problem description and controller design

2.1. The TS fuzzy model

A general TS fuzzy system[24] is described as follows.

Rule i: IF $x_1(k)$ is M_{i1}, $\cdots$, and $x_n(k)$ is M_{in}, THEN

$$x_{k+1} = A_i x_k + u_k, \tag{1}$$

where $x_k = [x_k(1), \cdots, \ x_k(n)]^{\mathrm{T}}$, $u_k = [u_k(1), \cdots, \ u_k(n)]^{\mathrm{T}}$, $i = 1, 2, \cdots, r$, r is the number of IF–THEN rules, M_{ij} are fuzzy sets, and the equation $x_{k+1} = A_i x_k + u_k$ is the output from the ith IF-THEN rule. Assume that $\|A_i\|_\infty < 1$, $i = 1, 2, \cdots, r$, then all A_i are $n \times n$ Schur stable.

Now, given a pair of (x_k, u_k), the final output of the fuzzy system is inferred from

$$x_{k+1} = \frac{\sum\limits_{i=1}^{r} w_i(k)\{A_i x_k + u_k\}}{\sum\limits_{i=1}^{r} w_i(k)}, \tag{2}$$

where $w_i(k) = \prod\limits_{j=1}^{n} M_{ij}(x_k(j))$, $M_{ij}(x_k(j))$ is the degree of membership of $x_k(j)$ in M_{ij}, with

$$\begin{cases} \sum\limits_{i=1}^{r} w_i(k) > 0 \\ w_i(k) \geq 0, \quad i = 1, 2, \cdots, r. \end{cases}$$

By using $h_i(k) = w_i(k) / \sum\limits_{i=1}^{r} w_i(k)$ instead of $w_i(k)$, Eq.(2) is rewritten as

$$x_{k+1} = \left[\sum_{i=1}^{r} h_i(k) A_i \right] x_k + u_k. \tag{3}$$

Note that

$$\begin{cases} \sum\limits_{i=1}^{r} h_i(k) = 1 \\ h_i(k) \geq 0, \quad i = 1, 2, \cdots, r, \end{cases} \tag{4}$$

in which $h_i(k)$ can be regarded as the normalized weight of the IF–THEN rules.

2.2. Problem description and controller design

The chaotification problem is to design a control input sequence, $\{u_k\}_{k=0}^{\infty}$ with an arbitrarily small magnitude, $\bar{u} > 0$, namely,

$$\|u_k\|_\infty \leq \bar{u}, \quad \text{for all} \ \ k = 1, \ 2, \ \cdots, \tag{5}$$

such that the controlled system (3) becomes chaotic in the sense of Devaney. For this purpose, in this paper we design the controller in the form of

$$u_k = [\beta\varphi(\sigma x_k^1), \ \cdots, \ \beta\varphi(\sigma x_k^n)]^{\mathrm{T}} \equiv \beta \Phi(\sigma x_k), \tag{6}$$

where $x_k = [x_k^1, \ x_k^2, \cdots, \ x_k^n]^{\mathrm{T}}$, σ, β are two positive constants and $\varphi : R \to R$ is the overflow nonlinear function of the 2's complement arithmetic defined by[22]

$$\varphi(y) = y - 2m\varepsilon \quad \text{for}$$
$$(2m-1)\varepsilon \leq y \leq (2m+1)\varepsilon, \ m = 0, \ \pm 1, \ \pm 2, \ \cdots \tag{7}$$

where ε is the maximum of the 2's complement arithmetic. Obviously, $|\varphi(y)| \leq \varepsilon$ for any $y \in R$, so that $\|u_k\| \leq \beta\varepsilon$, where ε can be chosen to be small enough such that $\|u_k\| \leq \beta\varepsilon \leq \bar{u}$, as required by condition (5). Similar to Lemma 1 in Ref.[14], we can obtain the following Lemma.

Lemma 1 (Boundedness Theorem) The state vector of the controlled system (3), under the control of the controller (6), is uniformly bounded by a constant, $\beta\varepsilon(1-N)^{-1}$, where the constant N is defined by $N = \max\limits_{1\leq i\leq r} \{ \|A_i\|_\infty \} < 1$.

Finally, it should be noted that in applying the overflow nonlinearlty of the 2's complement arithmetic, it is typical to identify ("paste") the extreme ε to another extreme value $-\varepsilon$ in the range of the map, thereby obtaining a continuous differentiable map on a circle (the "circle map" from the unit circle to the unit circle).[3,7] Here and throughout, we follow this convention.

3. Verification of chaos in the sense of Devaney

In this section, we apply the Shi–Chen theorem[23] to prove that the controlled system (3) with the decentralized state feedback controller (6) is chaotic in the sense of Devaney.

3.1. Chaos in the sense of Devaney: the Shi–Chen theorem[23]

Since Li and Yorke[16] first introduced a precise mathematical definition of chaos, there appeared several different definitions of chaos; some are stronger and some are weaker, depending on requirements in different problems. In 1987, Devaney[17] presented an explicit definition of chaos for the following systems:

$$x_{n+1} = F(x_n), \quad n \geq 0, \tag{8}$$

where $F : V \to V$ is a map and V is a subset of R^m, which is described as follows. A continuous map F is said to be chaotic on V (in the sense of Devaney) if

(i) F is transitive;

(ii) the periodic points of F are dense in V;

(iii) F has sensitive dependence on initial conditions.

By a relevant result given in Ref.[25], Properties (i) and (ii) together imply Property (iii). So, Property (iii) is redundant in the above definition.

Since results obtained in this paper are related to a symbolic dynamical system, for convenience, some results of symbolic dynamical system[17] are first reviewed.

Let

$$\sum\nolimits_2^+ := \{s = (s_0 s_1 s_2 \ldots) : s_j = 0 \quad \text{or} \quad 1\}$$

and define a distance between two points $s = (s_0 s_1 s_2 \ldots)$ and $t = (t_0 t_1 t_2 \ldots)$ in $\sum_2^+$ by

$$\rho(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}.$$

Lemma 2[23] $\left(\sum_2^+, \rho\right)$ is a complete, compact, totally disconnected, and perfect metric space.

The shift map $\alpha : \sum_2^+ \to \sum_2^+$ defined by $\alpha(s_0 s_1 s_2 \ldots) = (s_1 s_2 \ldots)$ is continuous. The dynamical system governed by α is called a symbolic dynamical system and it has the following properties:

Lemma 3[17]

(i) Card $\mathrm{Per}_n(\alpha) = 2^n$;

(ii) $\mathrm{Per}(\alpha)$ is dense in $\sum_2^+$;

(iii) there exists a dense orbit of α in $\sum_2^+$;

where Card $\mathrm{Per}_n(\alpha)$ denotes the number of periodic points of period n for α.

It is clear that Property (iii) implies that α is transitive. Hence, this symbolic dynamical system is chaotic in the sense of Devaney.

The following criterion of chaos for system (8) was established in Ref.[23], and it plays an important role in the rest of this paper.

Theorem (Shi–Chen)[23] Let (X, d) be a metric space and V_1, V_2 be two disjoint compact subsets of X. If the continuous map $F : V_1 \cup V_2 \to X$ satisfies

(1) $F(V_j) \supset V_1 \cup V_2$ for $j = 1, 2$;

(2) there exists a constant $\lambda_0 > 1$ such that

$$d(F(x), F(y)) \geq \lambda_0 d(x, y),$$

$$\forall x, y \in V_1 \quad \text{and} \quad \forall x, y \in V_2;$$

then there exists a Cantor set $\Lambda \in V_1 \cup V_2$ such that $F : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\alpha : \sum_2^+ \to \sum_2^+$. Consequently, F is chaotic on Λ in the sense of Devaney.

3.2. The proposed chaotification algorithm yields Devaney chaos

Theorem 1 Suppose that $h_i(k)$, $i = 1, \cdots, r$, are continuously differentiable in the neighbourhood of the fixed point, $x^* = 0$ of the controlled system (3) with the controller (6). Let

$$\left[\sum_{i=1}^{r} h_i(k) A_i\right] x = f(x) = [f_1(x), \ \cdots, \ f_n(x)]^{\mathrm{T}}$$

and $\|f'(x)\|_\infty \leq N < 1$. If

$$\sigma > \sigma^* = \frac{(N+1)(2+\theta)}{\theta\beta}, \tag{9}$$

where $\theta \in (0, 1)$, then the controlled TS fuzzy system (3) with the controller (6) is chaotic in the sense of Devaney.

Proof The controlled system (3) with the controller (6) is

$$x_{k+1} = \left[\sum_{i=1}^{r} h_i(k)A_i\right] x_k + \beta\,\Phi(\sigma x_k) \equiv F(x_k). \quad (10)$$

Let $F(x) = [F_1(x),\ F_2(x),\ \cdots,\ F_n(x)]^{\mathrm{T}}$, $X = R^n$, $d(\cdot) = \|\cdot\|_\infty$, and choose two disjoint closed rectangular regions $V_1 = [-\theta\sigma^{-1}\varepsilon b, \theta\sigma^{-1}\varepsilon b] \subset R^n$ and $V_2 = [(2-\theta)\sigma^{-1}\varepsilon b, (2+\theta)\sigma^{-1}\varepsilon b] \subset R^n$. Apparently, $(R^n, \|\cdot\|_\infty)$ is a complete metric space. Since $\|f'(x)\|_\infty \le N < 1$, then

$$|f_i(x)| \le N\|x\|_\infty. \quad (11)$$

Furthermore, by (9), we can get

$$F_j(x) = f_j(x) + \beta\varphi(-\theta\varepsilon) \le \theta\sigma^{-1}\varepsilon N - \theta\beta\varepsilon \le -\theta\sigma^{-1}\varepsilon, \quad \forall x \in V_1 \quad \text{with} \quad x_j = -\theta\sigma^{-1}\varepsilon;$$

$$F_j(x) = f_j(x) + \beta\varphi(\theta\varepsilon) \ge -\theta\sigma^{-1}\varepsilon N + \theta\beta\varepsilon \ge (2+\theta)\sigma^{-1}\varepsilon, \quad \forall x \in V_1 \quad \text{with} \quad x_j = \theta\sigma^{-1}\varepsilon;$$

$$F_j(x) \le (2+\theta)\sigma^{-1}\varepsilon N - \theta\beta\varepsilon \le -\theta\sigma^{-1}\varepsilon, \quad \forall x \in V_2 \quad \text{with} \quad x_j = (2-\theta)\sigma^{-1}\varepsilon;$$

$$F_j(x) \ge -(2+\theta)\sigma^{-1}\varepsilon N + \theta\beta\varepsilon \ge (2+\theta)\sigma^{-1}\varepsilon, \forall x \in V_2 \quad \text{with} \quad x_j = (2+\theta)\sigma^{-1}\varepsilon.$$

The above relations and continuity of F on V_1 and V_2, imply that for each $\sigma > \sigma^*$,

$$F(V_1) \supset [-\theta\sigma^{-1}\varepsilon b, (2+\theta)\sigma^{-1}\varepsilon b] \supset V_1 \cup V_2, F(V_2) \supset [-\theta\sigma^{-1}\varepsilon b, (2+\theta)\sigma^{-1}\varepsilon b] \supset V_1 \cup V_2. \quad (12)$$

On the other hand, for all $x, y \in V_1$ and for all $x, y \in V_2$, we have

$$\begin{aligned} &\|F(x) - F(y)\|_\infty \\ =&\|\,[f(x) - f(y)] + \beta[\Phi(\sigma x) - \Phi(\sigma y)]\|_\infty \\ =&\|\,[f'(\xi) + \beta\sigma I_{n\times n}](x-y)\|_\infty \\ \ge&\|[f'(\xi) + \beta\sigma I_{n\times n}]^{-1}\|_\infty^{-1}\|x-y\|_\infty, \end{aligned} \quad (13)$$

and

$$\begin{aligned} &\|[f'(\xi) + \beta\sigma I_{n\times n}]^{-1}\|_\infty \\ =&(\beta\sigma)^{-1}\|((\beta\sigma)^{-1}f'(\xi) + I)^{-1}\|_\infty \\ \le&\frac{1}{\beta\sigma(1 - \|(\beta\sigma)^{-1}f'(\xi)\|_\infty)} \\ \le&\frac{1}{\beta\sigma - N}. \end{aligned} \quad (14)$$

Therefore, from (9), (13) and (14), we can obtain

$$\begin{aligned} &\|F(x) - F(y)\|_\infty \\ =&\|\,[f(x) - f(y)] + \beta[\Phi(\sigma x) - \Phi(\sigma y)]\|_\infty \\ \ge&(\beta\sigma - N)\|x-y\|_\infty \\ \ge&\lambda_0\|x-y\|_\infty, \end{aligned}$$

where $\beta\sigma - N \ge \lambda_0 > 1$ for $\sigma > \sigma^*$. Since V_1 and V_2 are two disjoint compact subsets, for any $\sigma > \sigma^*$, all the conditions in Shi–Chen theorem[23] are satisfied and consequently the theorem follows from the Shi–Chen theorem.[23] This completes the proof.

Remark 1 It is surprising to note that σ^* in the Theorem 1 is independent of ε.

4. A simulation example

Consider a nonchaotic discrete-time TS fuzzy model, given by

Rule i IF x_k is M_i, THEN $X_{k+1} = \begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A_i \begin{bmatrix} x_k \\ y_k \end{bmatrix} + u_k$, $i = 1, 2$, where $A_1 = \begin{bmatrix} d & 0.1 \\ -0.3 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} -d & 0.1 \\ -0.3 & 0 \end{bmatrix}$, $x_k \in [-d,\ d]$, $d > 0$, and membership functions

$$M_1 = 0.5(1 - x_k/d), \quad M_2 = 0.5(1 + x_k/d).$$

The controlled TS fuzzy system is described by

$$X_{k+1} = \left[\sum_{i=1}^{2} h_i(k)A_i\right] X_k + \beta\Phi(\sigma X_k). \quad (15)$$

The magnitude of the control input is arbitrarily chosen to be $\bar{u} = 0.1$. In the simulation, we choose $\beta = 1$, $\varepsilon = 0.1$, $X_0 = [0.1,\ 0.1]^{\mathrm{T}}$, and $d = 0.7$. Since $N = \max\limits_{1\le i\le r}\{\|A_i\|_\infty\} = 0.8$, the uncontrolled TS fuzzy system (15) with $u_k = 0$ is stable. From Theorem 1, we know that the controlled TS fuzzy system (15) is chaotic if $\sigma\beta > 5.4$. When $\sigma = 6.0$, i.e. $\sigma\beta = 6.0$, the phase portrait and time wave diagrams are shown in Fig.1, respectively. From Fig.1, we can see that $\|u_k\|_\infty \le 0.1$ and the controlled TS fuzzy system (15) displays complex chaotic behaviours. The largest Lyapunov exponent of this controlled system (15) is calculated as $\lambda_{\max} = 2.5405$. These simulation results verify the theoretical analysis in this paper.

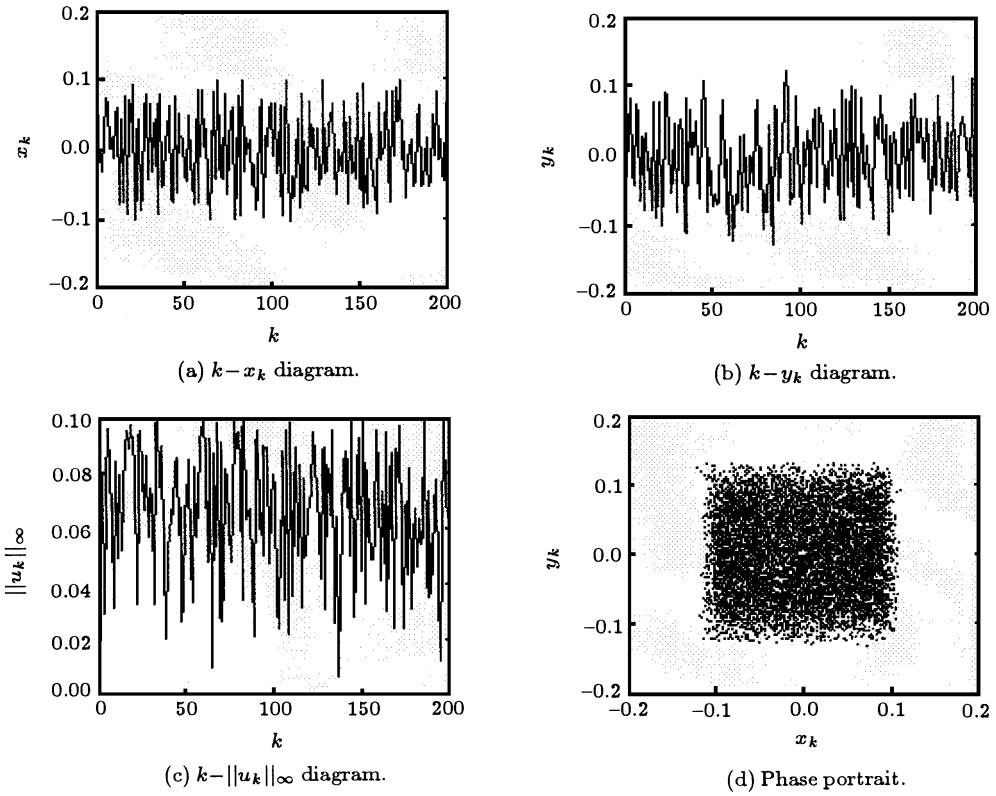


Fig.1. Simulation results.

5. Conclusions

The problem of making an originally stable TS fuzzy system chaotic in the sense of Devaney by using state feedback control of arbitrarily small magnitude has been studied. The feedback controller used is a decentralized linear feedback controller composed of the overflow nonlinear function of the 2's complement arithmetic. We have mathematically rigorously proved by using the Shi–Chen theorem that the controlled TS fuzzy system designed is indeed chaotic in the sense of Devaney. In particular, an explicit formula for the computation of control parameters has been obtained. Simulation results have been used to demonstrate the effectiveness of the theoretic results.

References

[1] Chen G and Fradkov A L Chaos Control and Synchronization Bibliographies (1987–2001), available via the WWW: http://www.ee.cityu.edu.hk/~gchen/chaos-papers.html
[2] Chen G and Dong X 1998 *From Chaos to Order—Perspectives, Methodologies, and Applications* (Singapore: World Scientific)
[3] Wang X F and Chen G 2000 *Int. J. Bifur. Chaos* **10** 549
[4] Chen G and Lai D 1996 *Int. J. Bifur. Chaos* **6** 1341
[5] Chen G and Lai D 1997 *IEEE Trans. Circuits Syst.* I **44** 250
[6] Chen G and Lai D 1998 *Int. J. Bifur. Chaos* **8** 1585
[7] Wang X F and Chen G 1999 *Int. J. Bifur. Chaos* **9** 1435
[8] Wang X F and Chen G 2000 *IEEE Trans. Circuit Syst.* I **47** 410
[9] Lu J G, Wang X F, Wei R and Wang Z Q 2000 *Proc. 19th Chinese Control Conference* 171
[10] Lu J G 2004 *Int. J. Nonlinear Sci. and Numer. Simul.* **5** 313
[11] Lu J G 2005 *Chin. Phys.* **14** 703
[12] Lu J G 2005 *Chin. Phys.* **14** 67
[13] Li Z *et al* 2002 *IEEE Trans. Circuit Syst.* I **49** 249
[14] Li Z *et al* 2002 *Int. J. Bifur. Chaos* **12** 2283
[15] Wei R and Wang X Y 2004 *Chin. Phys.* **13** 1210
[16] Li T Y and Yorke J A 1975 *Am. Math. Monthly* **82** 481
[17] Devaney R L 1987 *An Introduction to Chaotic Dynamical Systems* (New York: Addison-Wesley)
[18] Marotto F R 1978 *J. Math. Anal. Appl.* **63** 199
[19] Chen G, Hsu S B and Zhou J 1998 *J. Math. Phys.* **39** 6459
[20] Lin W, Ruan J and Zhao W R 2002 *Int. J. Bifur. Chaos* **12** 1129
[21] Li C P and Chen G 2003 *Chaos, Solitons Fractals* **18** 69
[22] Chua L O and Lin T 1988 *IEEE Trans. Circuit Syst.* **35** 648
[23] Shi Y and Chen G 2004 *Chaos, Solitons Fractals* **22** 555
[24] Takagi T and Sugeno M 1985 *IEEE Trans. Syst., Man, Cybern.* **15** 116
[25] Banks J, Brooks J, Cairns G, Davis G and Stacey P 1992 *Am. Math. Monthly* **99** 332