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A systematic approximate method for the study of evolution problem beyond rotating wave approximation

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By a model of a two-level particle coupled with boson field, we made it clear that an evolution problem can be solved beyond the rotating wave approximation. We have applied the coherent approximation method, which had been proved to be effective in dealing with stationary state problems of polaron, to the evolution problem of the system mentioned above. The results obtained showed that the coherent approximation method is effective to treat the evolution problem, and, in general cases, the non-rotating wave terms in Hamiltonian should not be ignored. Our results may provide a deep physical insight for further experiments to test the effects of non-rotating wave terms.

Keywords: coherent approximation method, non-rotating wave terms, Jaynes–Cummings model
PACC: 0365, 0290, 4250

1. Introduction

The Jaynes-Cummings model (JCM)[1−3] provides a useful means of studying quantum effects in the interaction between the electromagnetic field and matter. In this model the non-rotating wave terms in Hamiltonian are ignored. When the Rabi frequency, which is proportional to the intensity of the laser field, is small enough, the JCM is valid and has been successfully applied to many problems.[4−8] Recently, considerable interest has been paid to quantum information, quantum computing and quantum computer, etc,[9−12] where the intensity of a laser field is intense in many cases; thus it is necessity to take into account the effect of non-rotating wave terms in these problems.

In recent years, quite a few works have appeared suggesting different ways of improving the rotating wave approximation (RWA) or the approximations beyond JCM.[13−15] A typical problem is the real-time evolution of a two-level system with the action of a laser field, and its simplified model is just the JCM, in which the non-rotating wave terms are ignored. In order to improve RWA, we give a systematic approximation method in this paper, in which the non-rotating wave terms are included in Hamiltonian from the very beginning of the calculation.

When the non-rotating wave terms are included, the number of bosons is not conserved; therefore, using coherent state instead of Fock state is more suitable as Davydov first introduced it in these fields. To improve the original method introduced by Davydov in the polaron problem, we suggested the coherent approximation (CA) method, which provides a systematic expansion of coherent state, to calculate the ground state of polaron.[16−18] The CA method is a non-variation and non-perturbation method. And it is also a systematic and analytical approximation method. The key idea of the CA method is omitting the high-order terms of correlation effects, which contribute little to the system.[16] Furthermore, the CA method provides a clear physical view. With the CA method, we have obtained results much better than those from perturbation theory in the polaron problem.[16−18]

Encouraged by the success of application of CA to the stationary state problems in these fields, we try in this paper to treat the evolution problem mentioned above with the CA method. In order to test the va-

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lidity of CA in the dynamical processes, we calculate the evolution of JCM (with RWA) by CA method and compared the results with those by exact analytical formula derived directly from JCM. We found that they are in good agreement when suitable orders of approximation were taken in the expansion of the formula derived from CA method. Then, we show the necessities of including the non-rotating wave terms in the interaction Hamiltonian and the feasibility of CA method. We will see clearly in the following discussion, when the parameters in those kinds of problems are beyond the region of validity of RWA, the contributions from non-rotating wave terms are not negligible.

2. Coherent approximation

The Hamiltonian of the Rabi model, which describes the interaction between the boson field and a two-level system, has the following form:

$$\begin{aligned}\hat{H} =&\hat{H}_0 + \hat{V}\\ =&\hbar\omega_0|e\rangle\langle e| + \hbar\nu a^+a\\ &+ \hbar g(|e\rangle\langle g| + |g\rangle\langle e|)(a + a^+). \end{aligned} \tag{1}$$

Here $\hbar\omega_0$ is the energy of the excited state $|e\rangle$. The energy of the ground state $|g\rangle$ is zero. $\hbar\nu$ is the energy of a boson and g is a real parameter proportional to the interaction intensity.

Using the RWA, with the non-rotating wave terms ($|e\rangle\langle g|a^+ + |g\rangle\langle e|a$) ignored, we have the JCM Hamiltonian:

$$\begin{aligned}\hat{H} =&\hat{H}_0 + \hat{V}\\ =&\hbar\omega_0|e\rangle\langle e| + \hbar\nu a^+a\\ &+ \hbar g(|e\rangle\langle g|a + |g\rangle\langle e|a^+). \end{aligned} \tag{2}$$

CA is a systematic approximation method. For checking the validity of the CA applied to the systems represented by Eqs.(1) and (2), just like what was done with the state vectors of various orders in dealing with the stationary state problem of polaron in Ref.[16], we write the first three orders of the time-dependent state vectors as follows:

$$\begin{aligned}|t\rangle_1 =&\{[f_1(t) + g_1(t)a^+]|g\rangle\\ &+ [f_2(t) + g_2(t)a^+]|e\rangle\}|\rangle, \end{aligned} \tag{3}$$

$$\begin{aligned}|t\rangle_2 =&\{[f_1(t) + g_1(t)a^+ + h_1(t)a^+a^+]|g\rangle\\ &+ [f_2(t) + g_2(t)a^+ + h_2(t)a^+a^+]|e\rangle\}|\rangle, \end{aligned} \tag{4}$$

$$\begin{aligned}|t\rangle_3 =&\{[f_1(t) + g_1(t)a^+ + h_1(t)a^+a^+\\ &+ u_1(t)a^+a^+a^+]|g\rangle\\ &+ [f_2(t) + g_2(t)a^+ + h_2(t)a^+a^+\\ &+ u_2(t)a^+a^+a^+]|e\rangle\}|\rangle, \end{aligned} \tag{5}$$

where $|\rangle \equiv |\alpha(t)\rangle = \mathrm{e}^{\alpha(t)a^+}|0\rangle$ is the coherent state that satisfies $a|\rangle = \alpha(t)|\rangle$; $f_j(t)$, $g_j(t), h_j(t)$ and $u_j(t)$ are coefficient functions to be solved ($j = 1, 2$). After inserting them into the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|t\rangle = \hat{H}|t\rangle, \tag{6}$$

neglecting the corresponding higher-order terms in $(a^+)^n\mathrm{e}^{\alpha(t)a^+}|0\rangle$, we obtain the self-consistent coupled equations of coefficient functions, for example, in the first-order approximation we have the following equations.

For the Rabi model:

$$\mathrm{i}\dot{f}_1(t) = g[f_2(t)\alpha(t) + g_2(t)], \tag{7}$$

$$\begin{aligned}&\mathrm{i}\dot{g}_1(t) + \mathrm{i}\dot{\alpha}(t)f_1(t)\\ =&\nu[f_1(t)\alpha(t) + g_1(t)] + g[f_2(t) + g_2(t)\alpha(t)], \end{aligned} \tag{8}$$

$$\mathrm{i}\dot{f}_2(t) = \omega_0 f_2(t) + g[f_1(t)\alpha(t) + g_1(t)], \tag{9}$$

$$\begin{aligned}&\mathrm{i}\dot{g}_2(t) + \mathrm{i}\dot{\alpha}(t)f_2(t)\\ =&\omega_0 g_2(t) + \nu[f_2(t)\alpha(t) + g_2(t)]\\ &+ g[f_1(t) + g_1(t)\alpha(t)], \end{aligned} \tag{10}$$

$$g_1(t) + g_2(t) = 0; \tag{11}$$

For JCM:

$$\mathrm{i}\dot{f}_1(t) = 0, \tag{7a}$$

$$\mathrm{i}\dot{g}_1(t) + \mathrm{i}\dot{\alpha}(t)f_1(t) = \nu[f_1(t)\alpha(t) + g_1(t)] + g f_2(t), \tag{8a}$$

$$\mathrm{i}\dot{f}_2(t) = \omega_0 f_2(t) + g[f_1(t)\alpha(t) + g_1(t)], \tag{9a}$$

$$\begin{aligned}&\mathrm{i}\dot{g}_2(t) + \mathrm{i}\dot{\alpha}(t)f_2(t)\\ =&\omega_0 g_2(t) + \nu[f_2(t)\alpha(t) + g_2(t)] + g g_1(t)\alpha(t), \end{aligned} \tag{10a}$$

$$g_1(t) + g_2(t) = 0, \tag{11a}$$

where Eqs.(11) and (11a) are additional conditions to the coefficient functions to compensate one surplus coefficient function more than needed for the number of equations. Similar to the Lorentz condition added to Maxwell equations in electromagnetic field calculations, the conditions (11) and (11a) we take are physically reasonable and easy for calculation.

Using the coefficient functions obtained by solving the equations mentioned above, we are able to calculate the physical observables. To check the validity of the CA method, we calculate the occupancy probability of the ground state $p(t)$, which is one of the

most important physical observables of the system in the JCM, both by the analytical method and by the CA method under the same condition of RWA. Because the occupancy probability of the excited state is identically equal to $[1 - p(t)]$, we do not discuss it. The result obtained using the CA method, in the first order approximation, $p_{\mathrm{CA}-1}(t)$ has the following form:

$$\begin{aligned} p_{\mathrm{CA}-1}(t) =& \langle g|[f_1(t) + g_1(t)a^+]|t\rangle \\ =& \frac{|f_1(t)|^2 + |g_1(t)|^2(1 + |\alpha(t)|^2) + 2\mathrm{Re}[f_1(t)g_1^*(t)]}{\sum\limits_{j=1,2}\{[|f_j(t)|^2 + |g_j(t)|^2(1 + |\alpha(t)|^2)] + 2\mathrm{Re}[f_j(t)g_j^*(t)]\}}. \end{aligned} \tag{12}$$

And $p_{\mathrm{CA}-2}(t)$, $p_{\mathrm{CA}-3}(t)$ have the similar expressions.

As to the JCM, its Hamiltonian (2) allows us to gain the analytical solution. Concretely, we take the initial state of the system at $t = 0$ as $|g\rangle|\alpha_0\rangle$, i.e. the two-level system is at its ground state at $t = 0$, while the boson field is at the coherent state with a value α_0. From the exact analytical solution of the JCM, we have the occupancy probability of the ground state $p_{\mathrm{al}}(t)$:

$$p_{\mathrm{al}}(t) = \frac{1 + \sum\limits_{n=1}^{\infty}\frac{\alpha_0^{2n}}{n!}\left[\cos^2\left(\frac{1}{2}\sqrt{\delta^2 + 4ng^2}\,t\right) + \frac{\delta^2}{\delta^2 + 4ng^2}\sin^2\left(\frac{1}{2}\sqrt{\delta^2 + 4ng^2}\,t\right)\right]}{\exp(\alpha_0^2)}, \tag{13}$$

where $\delta = \nu - \omega_0$ is the resonance detuning of the boson field for the difference of energies of the two levels.

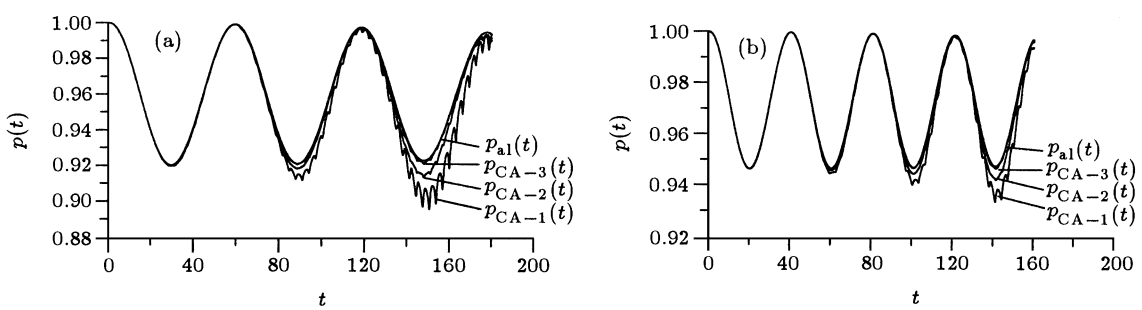


Fig.1. Comparison of the occupancy probabilities of the ground state $p(t)$ for JCM: $p_{\mathrm{al}}(t)$ obtained by analytical method and $p_{\mathrm{CA}-i}(t)$ by CA method; where $i = 1, 2, 3$ for 1st, 2nd and 3rd order approximations, respectively. (a) α=1.5, δ=0.1; (b) α=1.8, δ=0.15. Time is scaled by $\omega_0/2$.

Figure 1 shows the comparison of occupancy probabilities between $p_{\mathrm{al}}(t)$ from analytical method and $p_{\mathrm{CA}-i}(t)$ from CA method, where $i = 1, 2, 3$; and $p_{\mathrm{CA}-1}(t)$, $p_{\mathrm{CA}-2}(t)$, and $p_{\mathrm{CA}-3}(t)$ are its first, second and third order approximations, respectively. The parameters of the initial state in Fig.1 are:

$f_1(0) = 1.0$, and the others $g_j(0) = f_j(0) = g_j(0) = 0$.

From the figures we see that the results obtained by CA method are consistent with those from analytical solution. As it should be, the result of second-order approximation $p_{\mathrm{CA}-2}(t)$ is better than that of the first-order $p_{\mathrm{CA}-1}(t)$, and that of the third-order $p_{\mathrm{CA}-3}(t)$ is the best, which corresponds to the analytical result $p_{\mathrm{al}}(t)$. It should be emphasized that since the CA method is a systematic approximation, other initial parameter values, such as the intense interaction of laser field, the higher orders terms must be taken into account to ensure a better accuracy.

3. Comparison of NRWA with RWA by using the CA method

In this section, we compare the non-rotating wave approximation (NRWA, with non-rotating wave terms included) with the RWA by using the CA method.

Specifically, we consider the evolution of the number of phonons with time $N(t)$ in the Rabi model and JCM, i.e. we calculate $N(t)$ by the CA method with coefficient functions both from Eqs.(7)–(11) for the Rabi model and from Eqs.(7a)–(11a) for the JCM:

$$N(t) = \langle t|a^+ a|t\rangle = \frac{B_1(t) + B_2(t)}{D_1(t) + D_2(t)}, \qquad (14)$$

where, in the second-order approximation for the CA method, $B_j(t)$ and $D_j(t)$ are

$$\begin{aligned} B_1(t) = \sum_{j=1}^{2} [&|f_j(t)|^2 |\alpha(t)|^2 \\ &+ |g_j(t)|^2 (1 + 3|\alpha(t)|^2 + |\alpha(t)|^4) \\ &+ |h_j(t)|^2 (4 + 14|\alpha(t)|^2 \\ &+ 8|\alpha(t)|^4 + |\alpha(t)|^6)], \end{aligned} \qquad (15)$$

$$\begin{aligned} B_2(t) = \sum_{j=1}^{2} &2\mathrm{Re}[f_j(t) g_j^*(t)(1 + |\alpha(t)|^2)\alpha(t) \\ &+ f_j(t) h_j^*(t)(2 + |\alpha(t)|^2)(\alpha(t))^2 \\ &+ g_j(t) h_j^*(t)(4 + 5|\alpha(t)|^2 + |\alpha(t)|^4)\alpha(t)], \end{aligned} \qquad (16)$$

$$\begin{aligned} D_1(t) = \sum_{j=1}^{2} &[|f_j(t)|^2 + |g_j(t)|^2 (1 + |\alpha(t)|^2) \\ &+ |h_j(t)|^2 (2 + 4|\alpha(t)|^2 + |\alpha(t)|^4)], \end{aligned} \qquad (17)$$

$$\begin{aligned} D_2(t) = \sum_{j=1}^{2} &2\mathrm{Re}[f_j(t) g_j^*(t) + f_j(t) h_j^*(t)(\alpha(t))^2 \\ &+ g_j(t) h_j^*(t)(2 + |\alpha(t)|^2)\alpha(t)]. \end{aligned} \qquad (18)$$

For higher orders of approximation, Eq.(14) has similar expression. The results are shown in Fig.2.

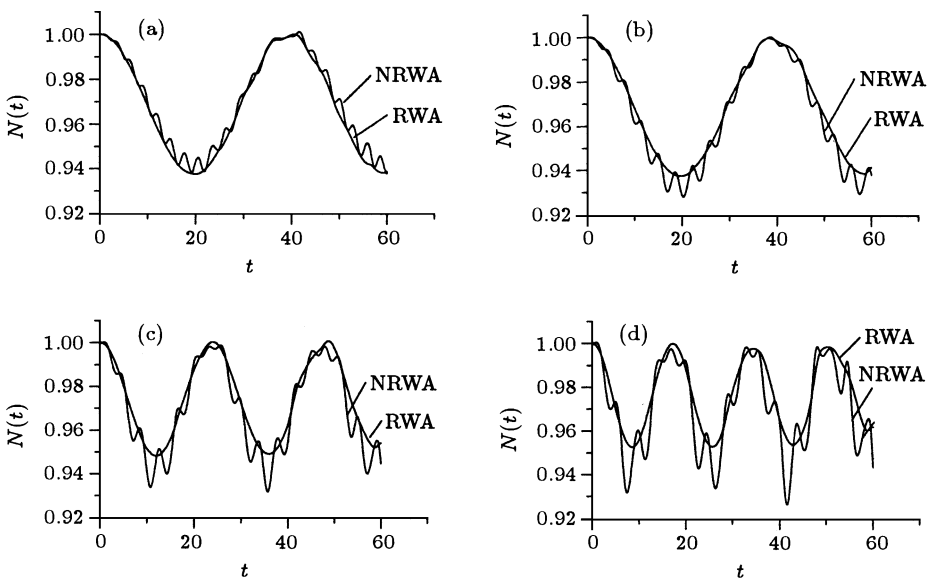


Fig.2. Comparison of the numbers of phonons $N(t)$ for RWA and NRWA calculated by the CA method; $\omega_0 = 1.0$, α_0=1.0; (a)$\delta = 0.15$, g=0.02; (b) $\delta = -0.15$, g=0.02; (c) $\delta = -0.25$, $g = 0.03$; (d) $\delta = -0.35$, $g = 0.04$. Time t is scaled by ω_0.

In Fig.2, the results of $N(t)$ both by using RWA and NRWA are compared in the fourth order approximation of the CA method. In the cases of small resonance detuning δ and small interaction intensity g, we have RWA (smooth) and NRWA (saw-tooth) curves as shown in Fig.2(a) ($\delta > 0$) and Fig.2(b) ($\delta < 0$). When the resonance detuning and the interaction intensity are comparatively large (Figs.2(c) and 2(d), $\delta < 0$), the difference becomes so remarkable that RWA is clearly no longer a good approximation, and the non-rotating wave terms in Hamiltonian must be considered.

We can have more clear information about RWA and NRWA by checking the occupancy probability of the ground state $p(t)$ in frequency space, i.e. we can transform $p(t)$ into frequency space $p(f)$ and compare the results of NRWA with those of RWA. With the same parameters as those in Fig.2, the results obtained are shown in Fig.3.

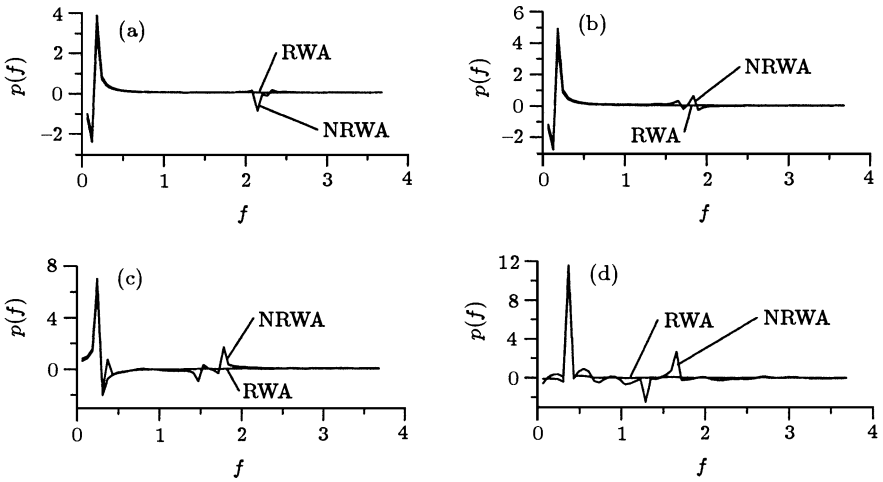


Fig.3. The occupancy probabilities of ground states in frequency space calculated by means of CA for RWA and NRWA. (a) $\delta = 0.15$, g=0.02; (b) $\delta = -0.15$, g=0.02; (c) $\delta = -0.25$,$g = 0.03$; (d) $\delta = -0.35$,$g = 0.04$.

Figures 3(a), 3(b), 3(c) and 3(d) are the frequency spectra of the occupancy probabilities of ground state $p(f)$. The main peaks in both curves of RWA and NRWA in the lower frequency area are the Rabi peaks corresponding to the Rabi oscillation in the JCM. It is clear that, besides the Rabi peak, there are NRW-peaks (non-rotating wave peaks) in the NRWA curves in higher frequency area, which are the contribution of the non-rotating wave terms in the Hamiltonian. The NRW-peaks become more and more important with the increase of the resonance detuning δ and the interaction intensity g. Although the intensities of NRW-peaks are smaller than those of the main Rabi-peaks, they are comparable in magnitudes and the curves of the NRWA have more complicated structures. Clearly, the effects of non-rotating wave terms should not be ignored in general cases. Recently, some works were reported in experimentally checking the Rabi oscillation.[19−21] Our results in this paper provide an insight for further experiments to test directly the Rabi oscillation and the effect of the non-rotating wave terms.

4. Conclusions

We have checked the applicability of the CA method in solving the evolution problem by a model of a two-level system coupled with a boson field.

First, the problem was solved under the rotating wave approximation (the simple JCM) by the CA method because the exact analytical solution of the JCM can be obtained. Comparing the solutions both by the CA method and by the exact analytical method, we found that, with higher order approximation terms in CA involved, the results of CA may be obtained very near the exact analytical solution.

Then, the contribution of the non-rotating wave terms in the problem were studied by the CA method. A comparison between the results from the rotating wave approximation (RWA) and those from the non-rotating wave approximation (NRWA), i.e. between the JCM and the Rabi model, was carried out. When the resonance detuning and interaction intensity are small, the evolution curves of phonon number expectation value for both models showed a good correspondence in behaviour. The non-rotating wave terms add small contributions in the form of saw-tooth wave to the smooth curve of the JCM. However, when the resonance detuning and interaction intensity increase, the behaviours of both models are quite difference from each other, i.e. RWA is no longer valid, and the non-rotating wave terms must be taken into account. When the occupancy probabilities of ground state for both RWA and NRWA are compared in frequency space, the contributions of the non-rotating wave terms are appreciated in the form of a series of high-frequency peaks, which may provide a deep physical insight for further experiments to test the Rabi oscillation and the effects of the non-rotating wave terms in the Hamiltonian.

As the CA method is a non-variation and non-perturbation method, it is not restricted by the value of parameters, and it is much more suitable when the coupling parameters are increased. In addition, because of the systematic and analytical characters, the

CA method is more concise and can show more clear physical insight than pure numerical calculation. So, the CA method has shown itself an to be effective tool in treating evolution problems.

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