Vol 14 No 7, July 2005
1009-1963/2005/14(07)/1307-06

Chinese Physics

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Application of new entangled state representation to a two-coupled oscillator system

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(Received 9 October 2004; revised manuscript received 4 March 2005)

We study the eigenstate problem of a two-coupled oscillator system. A new entangled state representation $|\gamma\rangle$ composed of the common eigenvectors of operators (x_1+p_2) and (p_1+x_2) is established. Eigenvalues and eigenvectors of the Hamiltonian are obtained in $|\gamma\rangle$ representations. The same problem is studied in the second-quantization representation. We find that the second-quantization representation can be used to derive the normally ordered product expression of eigenvector in Fock space. In particular, we find that the ground state of the Hamiltonian is a kind of generalized two-mode squeezed state.

Keywords: entangled state, two coupled oscillator system
PACC: 0365

1. Introduction

It is well known that Dirac[1] established quantum mechanical representation theory that played a very important role in the development of modern quantum physics. We know that the coherent state representation[2,3] and the entangled state representation[3,4] have greatly promoted the development of quantum optics, quantum information and computation.[5,6] By using the technique of Integration Within Ordered Product of operators (IWOP technique), Fan[3,7] has developed Dirac's representation theory and symbolic method.

In this paper, we study the eigenstate problem of a two-coupled oscillator system. We first construct a new entangled state representation $|\gamma\rangle$ composed of the common eigenvectors of operators (x_1+p_2) and (p_1+x_2), where x_l and p_l are respectively the position and momentum operators of the lth particle $(l=1,2)$. Eigenvalues and eigenvectors of the Hamiltonian are obtained in the $|\gamma\rangle$ representation. We find that the $|\gamma\rangle$ representations give a good picture for working out this problem. The same problem is studied in the second-quantization representation. We find that the second-quantization representation can be used to derive the normally ordered product expression of eigenvector in Fock space. In particular, we find that the ground state of the Hamiltonian is a generalized two-mode squeezed state.

This paper is arranged as follows. In Section 2, we construct the $|\gamma\rangle$ representation. In Section 3, we solve the eigenstate problem in the $|\gamma\rangle$ representation. In Section 4, the normally ordered product expression of eigenvector in Fock space is presented.

2. Construction of the new entangled state representation $|\gamma\rangle$

We define the wavefunction

$$|\gamma\rangle = \exp\left\{-\frac{|\gamma|^2}{2}+\gamma a_1^\dagger+\mathrm{i}\gamma^* a_2^+-\mathrm{i}a_1^\dagger a_2^+\right\}|0,0\rangle,$$
$$\gamma = \gamma_1+\mathrm{i}\gamma_2. \tag{1}$$

It is easy to prove that $|\gamma\rangle$ is the simultaneous eigenvector of operators (x_1+p_2) and (p_1+x_2) as follows:

$$(x_1+p_2)|\gamma\rangle = \sqrt{2}\gamma_1|\gamma\rangle,$$
$$(p_1+x_2)|\gamma\rangle = \sqrt{2}\gamma_2|\gamma\rangle, \tag{2}$$

where $x_l=\frac{1}{\sqrt{2}}(a_l+a_l^\dagger)$ and $p_l=\frac{1}{\sqrt{2}\mathrm{i}}(a_l-a_l^\dagger)$ are respectively the position and momentum operators of the lth particle $(l=1,2)$. When the operators (x_1-p_2) and (p_1-x_2) respectively act on $|\gamma\rangle$, we

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obtain two important formulas as follows:

$$\langle\gamma|(p_1 - x_2) = -\mathrm{i}\sqrt{2}\frac{\partial}{\partial\gamma_1}\langle\gamma|,$$
$$\langle\gamma|(x_1 - p_2) = \mathrm{i}\sqrt{2}\frac{\partial}{\partial\gamma_2}\langle\gamma|. \quad (3)$$

The expressions (2) and (3) will be used in the next section. Because of $\gamma = \gamma_1 + \mathrm{i}\gamma_2$, the expression (1) can be rewritten as

$$|\gamma\rangle = \exp\Big\{ -\frac{\gamma_1^2}{2} - \frac{\gamma_2^2}{2} + \gamma_1(a_1^\dagger + \mathrm{i}a_2^\dagger) + \gamma_2(a_2^\dagger + \mathrm{i}a_1^\dagger) - \mathrm{i}a_1^\dagger a_2^\dagger\Big\}|0,0\rangle. \quad (4)$$

With the aid of IWOP technique, we can perform the integration over γ_1 and γ_2

$$\begin{aligned}&\int\frac{\mathrm{d}^2\gamma}{\pi}|\gamma\rangle\langle\gamma| \\ =&\int\frac{\mathrm{d}\gamma_1\mathrm{d}\gamma_2}{\pi} : \exp\{-\gamma_1^2 - \gamma_2^2 \\ &+ \gamma_1(a_1^\dagger + \mathrm{i}a_2^\dagger + a_1 - \mathrm{i}a_2)\} \\ &\times \exp\{\gamma_2(a_2^\dagger + \mathrm{i}a_1^\dagger + a_2 - \mathrm{i}a_1) \\ &- \mathrm{i}a_1^\dagger a_2^\dagger + \mathrm{i}a_1 a_2 - a_1^\dagger a_1 - a_2^\dagger a_2\} := 1, \end{aligned} \quad (5)$$

where the following formula[8] is used:

$$|0,0\rangle\langle 0,0| =: \exp\{-a_1^\dagger a_1 - a_2^\dagger a_2\} : \quad (6)$$

The symbol ": :" in the expression (6) refers to the normally ordered product of operator. In terms of the expression (5), the completeness relation of $|\gamma\rangle$ is proved. The orthogonality relation of $|\gamma\rangle$ can be proved also by using IWOP technique in coherent state representation. Here we discuss the problem in the other way. We construct two pairs of new creation and annihilation operators as follows:

$$\begin{aligned}C &= \frac{1}{\sqrt{2}}(a_1 + \mathrm{i}a_2), \\ C^\dagger &= \frac{1}{\sqrt{2}}(a_1^\dagger - \mathrm{i}a_2^\dagger), \\ D &= \frac{1}{\sqrt{2}}(a_1 - \mathrm{i}a_2), \\ D^\dagger &= \frac{1}{\sqrt{2}}(a_1^\dagger + \mathrm{i}a_2^\dagger). \end{aligned} \quad (7)$$

They satisfy the commutative relations

$$[C, C^\dagger] = [D, D^\dagger] = 1, \quad [C, D] = [C, D^\dagger] = 0. \quad (8)$$

From the expression (7), it is easy to prove that

$$|\gamma\rangle = \pi|q\rangle_{q=\gamma_1} \otimes |p\rangle_{p=\gamma_2}, \quad (9)$$

where $\otimes$ stands for the direct product of $|q\rangle$ and $|p\rangle$. They are defined as

$$\begin{aligned}|q\rangle &= \left(\frac{1}{\pi}\right)^{\frac{1}{4}}\exp\Big\{-\frac{q^2}{2} + \sqrt{2}qD^\dagger - \frac{D^{\dagger^2}}{2}\Big\}|0\rangle_D, \\ |p\rangle &= \left(\frac{1}{\pi}\right)^{\frac{1}{4}}\exp\Big\{-\frac{p^2}{2} + \sqrt{2}\mathrm{i}pC^\dagger + \frac{C^{\dagger^2}}{2}\Big\}|0\rangle_C . \end{aligned} \quad (10)$$

The vacuum states $|0\rangle_D$ and $|0\rangle_C$ are determined by

$$C|0,0\rangle_{C,D} = D\ |0,0\rangle_{C,D} = 0, \quad (11)$$

where

$$|0,0\rangle_{C,D} = |0\rangle_C\ \otimes|0\rangle_D. \quad (12)$$

From the expressions (8) and (10), we can obtain

$$\langle q'|q''\rangle = \delta(q' - q'')$$

and

$$\langle p'|p''\rangle = \delta(p' - p''). \quad (13)$$

From the expressions (9) and (13), we have

$$\begin{aligned}\langle\gamma'|\gamma''\rangle =&\pi_{q'=\gamma_1'}\langle q'|q''\rangle_{q''=\gamma_1''}\otimes_{p'=\gamma_2'}\langle p'|p''\rangle_{p''=\gamma_2''} \\ =&\pi\delta(\gamma_1' - \gamma_1'')\delta(\gamma_2' - \gamma_2''). \end{aligned} \quad (14)$$

This is just the orthogonality relation of eigenvectors $|\gamma\rangle$. It can be seen from the expressions (5) and (14) that the state $|\gamma\rangle$ constructs a quantum mechanical representation indeed. According to the expressions (A4) and (A6), we see that the representation $|\gamma\rangle$ is also an entangled state representation. The properties of the entangled state $|\gamma\rangle$ will be detailed in another work.

3. Solution to the eigenstate problem of the Hamiltonian in $|\gamma\rangle$ representation

In this section, we solve the eigenstate problem in the above discussed $|\gamma\rangle$ representation. The Hamiltonian of a two-coupled oscillator system is defined as

$$\begin{aligned}H =&\frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) \\ &+ k(x_1p_2 + x_2p_1). \end{aligned} \quad (15)$$

For simplicity, we assume that $m_1 = m_2 = 1$, $\omega_1 = \omega_2 = 1$ and $\hbar = 1$. We change the Hamiltonian into the following form:

$$H = \left[\frac{1}{2}(1 + \alpha^2)\right]^{-1}H', \quad (16)$$

where

$$H' = \left[\frac{1}{4}(p_1 + x_2)^2 + \frac{\alpha^2}{4}(x_2 - p_1)^2\right] + \left[\frac{1}{4}(p_2 + x_1)^2 + \frac{\alpha^2}{4}(x_1 - p_2)^2\right] \quad (17)$$

and

$$\alpha^2 = \frac{1-k}{1+k}, \qquad k = \frac{1-\alpha^2}{1+\alpha^2}. \quad (18)$$

We call $|E\rangle$ the eigenvector of the H'. Of course, $|E\rangle$ is also the eigenvector of the H. Then we have

$$H'|E\rangle = E'|E\rangle \text{ and } \langle\gamma|H'|E\rangle = E'\langle\gamma|E\rangle. \quad (19)$$

We define $\langle\gamma|E\rangle$ as

$$\langle\gamma|E\rangle = \langle\gamma_1\gamma_2|E\rangle = \varphi(\gamma_1, \gamma_2). \quad (20)$$

From the expressions (2),(3) and (20), the eigenequation (19) is changed into a differential equation

$$\left[\left(-\frac{\alpha^2}{2}\frac{\partial^2}{\partial\gamma_1^2} + \frac{\gamma_1^2}{2}\right) + \left(-\frac{\alpha^2}{2}\frac{\partial^2}{\partial\gamma_2^2} + \frac{\gamma_2^2}{2}\right)\right]\varphi(\gamma_1, \gamma_2) = E'\varphi(\gamma_1, \gamma_2). \quad (21)$$

The equation (21) is a variable separable differential equation. By setting

$$\varphi(r_1 r_2) = \varphi_1(r_1)\varphi_2(r_2), \quad E' = E_1' + E_2' \quad (22)$$

and

$$\lambda_1 = \frac{2E_1'}{\alpha}, \quad \lambda_2 = \frac{2E_2'}{\alpha}, \quad \xi_1 = \frac{\gamma_1}{\sqrt{\alpha}}, \quad \xi_2 = \frac{\gamma_2}{\sqrt{\alpha}}, \quad (23)$$

we can divide Eq.(21) into two parts as follows:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi_1^2} + (\lambda_1 - \xi_1^2)\right]\varphi_1(\xi_1) = 0, \quad \left[\frac{\mathrm{d}^2}{\mathrm{d}\xi_2^2} + (\lambda_1 - \xi_2^2)\right]\varphi_2(\xi_2) = 0. \quad (24)$$

They are the equations that can be changed into standard Hermite equations. Then the eigenvalue E' of H' is directly obtained as follows:

$$E' = E_1' + E_2' = \alpha(n_1 + n_2 + 1), \quad (25)$$

where $n_1, n_2 = 0, 1, 2, \ldots$. From the expression (16), the eigenvalue of the Hamiltonian H is

$$E = \frac{2\alpha}{1+\alpha^2}(n_1 + n_2 + 1) = \sqrt{1-k^2}(n_1 + n_2 + 1). \quad (26)$$

And we have the eigenvector of the Hamiltonian H in the function space

$$\langle\gamma_1, \gamma_2|E\rangle = N_{n_1 n_2}\mathrm{e}^{-\frac{1}{2\alpha}(\gamma_1^2 + \gamma_2^2)} \times H_{n_1}\left(\frac{\gamma_1}{\sqrt{\alpha}}\right)H_{n_2}\left(\frac{\gamma_2}{\sqrt{\alpha}}\right), \quad (27)$$

where

$$N_{n_1 n_2} = (2^{n_1+n_2} n_1! n_2! \pi)^{-\frac{1}{2}}. \quad (28)$$

4. The normally ordered product expression of eigenvectors in Fock space and the squeezed state

We derive the Hamiltonian expression in the second-quantization representation, thereby we can obtain the normally ordered product expression of the eigenvectors in Fock space. We rewrite expression (17) as

$$H' = \frac{P'^2}{2} + \frac{\alpha^2}{2}P''^2 + \frac{\alpha^2}{2}Q'^2 + \frac{Q''^2}{2}, \quad (29)$$

where

$$Q' = \frac{1}{\sqrt{2}}(x_1 - p_2), \quad P' = \frac{1}{\sqrt{2}}(p_1 + x_2), \quad Q'' = \frac{1}{\sqrt{2}}(x_1 + p_2), \quad P'' = \frac{1}{\sqrt{2}}(p_1 - x_2). \quad (30)$$

They have the following commutative relations:

$$[Q', P'] = [Q'', P''] = \mathrm{i}, \quad [Q', Q''] = [Q', P''] = [P', Q''] = [P', P''] = 0. \quad (31)$$

We define operators

$$A = \frac{1}{\sqrt{2}}(\sqrt{\alpha}Q' + \frac{\mathrm{i}}{\sqrt{\alpha}}P'), \quad A^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\alpha}Q' - \frac{\mathrm{i}}{\sqrt{\alpha}}P'), \quad B = \frac{1}{\sqrt{2}}(\frac{Q''}{\sqrt{\alpha}} + \mathrm{i}\sqrt{\alpha}P''), \quad B^\dagger = \frac{1}{\sqrt{2}}(\frac{Q''}{\sqrt{\alpha}} - \mathrm{i}\sqrt{\alpha}P''), \quad (32)$$

They satisfy the commutative relations:

$$[A, A^\dagger] = [B, B^\dagger] = 1, \ [A, B] = [A, B^\dagger] = 0\ . \quad (33)$$

It is easy to prove that the Hamiltonian (29) can be expressed in $(A, A^{\dagger})$ and $(B, B^{\dagger})$ as

$$H' = \alpha\left(A^{\dagger}A + \frac{1}{2}\right) + \alpha\left(B^{\dagger}B + \frac{1}{2}\right), \tag{34}$$

which is just the expression of H' in the second-quantization representation. From the expression (34), we directly obtain the eigenvalue of Hamiltonian H'

$$E' = \alpha(n_1 + n_2 + 1), \tag{35}$$

which is the same as the expression (25). It is the eigenvector of the Hamiltonian that is the eigenvector of a generalized two-mode oscillator

$$\Psi_{n_1,n_2} = \frac{1}{\sqrt{n_1!n_2!}} A^{\dagger n_1} B^{\dagger n_2} |0,0\rangle_{A,B}, \tag{36}$$

where the operators $(A^{\dagger}, B^{\dagger})$ are defined in the expression (32). And it is the vacuum state $|0,0\rangle_{A,B}$ determined by

$$A|0,0\rangle_{A,B} = 0, \quad B|0,0\rangle_{A,B} = 0 \tag{37}$$

that is the ground state of the Hamiltonian. Based on the expression (37), we can derive the normally ordered product expression of the eigenvector in Fock space. From the expressions (30) and (32), we rewrite the operators A and B respectively as

$$A = \nu C + \mu C^{\dagger}, \quad B = \nu D - \mu D^{\dagger}, \tag{38}$$

where the operators C and D are expressed by the expression (7),

$$\nu = \frac{1}{2\sqrt{\alpha}}(\alpha + 1), \quad \mu = \frac{1}{2\sqrt{\alpha}}(\alpha - 1). \tag{39}$$

The relation between ν and μ is as follows:

$$\nu^2 - \mu^2 = 1. \tag{40}$$

Using the expressions (37) and (38), we have

$$\nu C|0,0\rangle_{A,B} = -\mu C^{\dagger}|0,0\rangle_{A,B}$$

and

$$\nu D|0,0\rangle_{A,B} = \mu D^{\dagger}|0,0\rangle_{A,B}. \tag{41}$$

Then we have

$$|0,0\rangle_{A,B} = U|0,0\rangle_{C,D}$$

and

$$U = \exp\left[-\frac{\mu}{2\nu}(C^{\dagger 2} - D^{\dagger 2})\right]. \tag{42}$$

The empty state $|0,0\rangle_{C,D}$ in the expression (42) is defined by the expression (11). From the expressions (11) and (7), we also have

$$a_1|0,0\rangle_{C,D} = a_2|0,0\rangle_{C,D} = 0. \tag{43}$$

Using the expressions (11) and (43), we have

$$|0,0\rangle_{C,D} \equiv |0,0\rangle_{a_1,a_2}. \tag{44}$$

Using the expression (42), we can change the eigenvector (36) into

$$\begin{aligned}\psi_{n_1n_2} =& \frac{1}{\sqrt{n_1!n_2!}} A^{\dagger n_1} B^{\dagger n_2}|0,0\rangle_{A,B} \\ =& \frac{1}{\sqrt{n_1!n_2!}} U(U^{-1}A^{\dagger}U)^{n_1} \\ & \times (U^{-1}B^{\dagger}U)^{n_2}|0,0\rangle_{C,D}.\end{aligned} \tag{45}$$

By using the following operator formula:

$$\mathrm{e}^{\widehat{F}}\widehat{G}\mathrm{e}^{\widehat{-F}} = \widehat{G} + \widehat{[F,G]} + \frac{1}{2!}[\widehat{F}\ \widehat{[F,G]}]\ + ..., \tag{46}$$

we obtain

$$\begin{aligned}U^{-1}A^{\dagger}U &= \mu C + \frac{C^{\dagger}}{\nu}, \\ U^{-1}B^{\dagger}U &= -\mu D + \frac{D^{\dagger}}{\nu}.\end{aligned} \tag{47}$$

Substituting the expression (47) into the expression (45) yields

$$\begin{aligned}\psi_{n_1n_2} =& \frac{1}{\sqrt{n_1!n_2!}} \exp\left[-\frac{\mu}{2\nu}(C^{\dagger 2} - D^{\dagger 2})\right] \\ & \times \left(\nu C + \frac{C^{\dagger}}{\nu}\right)^{n_1} \\ & \times \left(-\mu D + \frac{D^{\dagger}}{\nu}\right)^{n_2}|0,0\rangle_{C,D}\ .\end{aligned} \tag{48}$$

In order to obtain the normally ordered product expression (48), we need to make $\left(\nu C + \frac{C^{\dagger}}{\nu}\right)^{n_1} \times \left(-\mu D + \frac{D^{\dagger}}{\nu}\right)^{n_2}$ normally ordered. Fortunately, there exists a useful operator formula

$$\begin{aligned}(sa + ta^{\dagger})^n =& \left(-\mathrm{i}\sqrt{\frac{st}{2}}\right)^n \\ & \times :H_n\left(\mathrm{i}\sqrt{\frac{s}{2t}}a + \mathrm{i}\sqrt{\frac{t}{2s}}a^{\dagger}\right):\end{aligned} \tag{49}$$

where s and t are two real parameters. The formula (49) has been proved with quantum similarity transformation and IWOP technique of operator.[3,10] Then

we have

$$\begin{aligned}
&\left(\nu C+\frac{C^{\dagger}}{\nu}\right)^{n_1}\left(-\mu D+\frac{D^{\dagger}}{\nu}\right)^{n_2}|0,0\rangle_{C,D}\\
=&\left(-\mathrm{i}\sqrt{\frac{\mu}{2\nu}}\right)^{n_1}\left(-\mathrm{i}\sqrt{\frac{-\mu}{2\nu}}\right)^{n_2}\\
&\times:H_{n_1}\left(\mathrm{i}\sqrt{\frac{\mu\nu}{2}}C+\mathrm{i}\sqrt{\frac{1}{2\mu\nu}}C^{\dagger}\right):\\
&\times:H_{n_2}\left(\mathrm{i}\sqrt{\frac{-\mu\nu}{2}}D+\mathrm{i}\sqrt{\frac{-1}{2\mu\nu}}D^{\dagger}\right):|0,0\rangle_{C,D}\\
=&\left(-\mathrm{i}\sqrt{\frac{\mu}{2\nu}}\right)^{n_1}\left(\sqrt{\frac{\mu}{2\nu}}\right)^{n_2}\\
&\times H_{n_1}\left(\mathrm{i}\sqrt{\frac{1}{2\mu\nu}}C^{\dagger}\right)\\
&\times H_{n_2}\left(-\sqrt{\frac{1}{2\mu\nu}}D^{\dagger}\right)|0,0\rangle_{C,D}. \qquad (50)
\end{aligned}$$

Using the expressions (7), (39), (48) and (50), we obtain the normally ordered product expression of eigenvector in Fock space as follows:

$$\begin{aligned}
\psi_{n_1n_2}=&\frac{1}{\sqrt{n_1!n_2!}}(-\mathrm{i})^{n_1}\left(\sqrt{\frac{\alpha-1}{2(\alpha+1)}}\right)^{n_1+n_2}\\
&\times\exp\left(\mathrm{i}\frac{\alpha-1}{\alpha+1}a_1^{\dagger}a_2^{\dagger}\right)\cdot\\
&\times H_{n_1}\left[\mathrm{i}\sqrt{\frac{\alpha}{\alpha^2-1}}(a_1^{\dagger}-\mathrm{i}a_2^{\dagger})\right]\\
&\times H_{n_2}\left[-\sqrt{\frac{\alpha}{\alpha^2-1}}(a_1^{\dagger}+\mathrm{i}a_2^{\dagger})\right]|0,0\rangle_{a_1,a_2}, \qquad (51)
\end{aligned}$$

where α is defined by the expression (4). From the expression (51), we obtain the ground state of the Hamiltonian

$$\psi_{0,0}=\exp\left(\mathrm{i}\frac{\alpha-1}{\alpha+1}a_1^{\dagger}a_2^{\dagger}\right)|0,0\rangle_{a_1,a_2}, \qquad (52)$$

which is a generalized two-mode squeezed state.

5. Conclusion

We have studied the eigenstate problem of a two-coupled oscillator system. We have constructed a new entangled state representation $|\gamma\rangle$ composed of the common eigenvectors of operators (x_1+p_2) and (p_1+x_2). Eigenvalues and eigenvectors of the Hamiltonian are obtained in $|\gamma\rangle$ representation. The same problem is also studied in the second-quantization representation. The second-quantization representation can be used to derive the normally ordered product expression of eigenvector in Fock space. The ground state of the Hamiltonian is found to be a generalized two-mode squeezed state.

Appendix: $|\gamma\rangle$ state is a new entangled state

In Section 3, we have proved that the state $|\gamma\rangle$ constructs a quantum mechanical representation. In this Appendix, we prove that the $|\gamma\rangle$ state is a new entangled state . It is known that

$$|\gamma\rangle=\exp\left\{-\frac{|\gamma|^2}{2}+\gamma a_1^{\dagger}+\mathrm{i}\gamma^{*}a_2^{\dagger}-\mathrm{i}a_1^{\dagger}a_2^{\dagger}\right\}|0,0\rangle. \qquad (A1)$$

Because of $\gamma=\gamma_1+\gamma_2$, $|\gamma\rangle$ can be rewritten as

$$\begin{aligned}
|\gamma\rangle=&\exp\left\{-\frac{\gamma_1^2}{2}-\frac{\gamma_2^2}{2}+\gamma_1(a_1^{\dagger}+\mathrm{i}a_2^{\dagger})\right.\\
&\left.+\gamma_2(a_2^{\dagger}+\mathrm{i}a_1^{\dagger})-\mathrm{i}a_1^{\dagger}a_2^{\dagger}\right\}|0,0\rangle. \qquad (A2)
\end{aligned}$$

We make a Fourier transformation to the state $|\gamma\rangle$ as follows:

$$\begin{aligned}
&\int\frac{\mathrm{d}\gamma_2}{2\pi}\left|\gamma=\frac{1}{\sqrt{2}}(\gamma_1+\mathrm{i}\gamma_2)\right\rangle\mathrm{e}^{-\mathrm{i}u\gamma_2}\\
=&\left|u+\frac{\gamma_1}{2}\right\rangle_{a_1}^{q}\otimes\left|-\left(u-\frac{\gamma_1}{2}\right)\right\rangle_{a_2}^{p}, \qquad (A3)
\end{aligned}$$

where $\left|u+\frac{\gamma_1}{2}\right\rangle_{a_1^{\dagger}}^{q}$ is the eigenvector of position operator for mode 1 which is marked by q and a_1, and $\left|-\left(u-\frac{\gamma_1}{2}\right)\right\rangle_{a_2}^{p}$ is the eigenvector of momentum operator for mode 2 which is marked by p and a_2. From the expression (A3), it is easy to obtain the inverse Fourier transformation as follows:

$$\begin{aligned}
&\left|\gamma=\frac{1}{\sqrt{2}}(\gamma_1+\mathrm{i}\gamma_2)\right\rangle\\
=&\mathrm{e}^{-\mathrm{i}\frac{\gamma_1\gamma_2}{2}}\int_{-\infty}^{\infty}\mathrm{d}u|u\rangle_{a_1}^{q}\otimes|\gamma_1-u\rangle_{a_2}^{p}\mathrm{e}^{\mathrm{i}u\gamma_2}. \qquad (A4)
\end{aligned}$$

This is just the Schmidt decomposition of the state $|\gamma\rangle$. Just like the derivation of the expressions (A3) and (A4), we can also have

$$\begin{aligned}
&\int\frac{\mathrm{d}\gamma_1}{2\pi}\left|\gamma=\frac{1}{\sqrt{2}}(\gamma_1+\mathrm{i}\gamma_2)\right\rangle\mathrm{e}^{\mathrm{i}v\gamma_1}\\
=&\left|v+\frac{\gamma_2}{2}\right\rangle_{a_1}^{p}\otimes\left|-v+\frac{\gamma_2}{2}\right\rangle_{a_2}^{q}, \qquad (A5)
\end{aligned}$$

$$\begin{aligned}
&\left|\gamma=\frac{1}{\sqrt{2}}(\gamma_1+\mathrm{i}\gamma_2)\right\rangle\\
=&\mathrm{e}^{-\mathrm{i}\frac{\gamma_1\gamma_2}{2}}\int_{-\infty}^{\infty}\mathrm{d}v|v+\gamma_2\rangle_{a_1}^{p}\otimes|-v\rangle_{a_2}^{q}\mathrm{e}^{-\mathrm{i}v\gamma_1}. \qquad (A6)
\end{aligned}$$

By use of the expression (A4) or (A6), the state $|\gamma\rangle$ is really indicated to be a new entangled state.

References

[1] Dirac P A M 1958 *The Principles of Quantum Mechanics* (Oxford: Oxford University Press)

[2] Klauder J R and Skargerstam B S 1985 *Coherent State* (Singapore: World Scientific)
Perelomov A 1986 *Generalized Coherent State and Their Application* (Berlin: Springer)

[3] Fan H Y 1997 *Representation and Transformation Theory in Quantum Mechanics* (Shanghai: Shanghai Scientific and Technical Publishers) (in Chinese)
Fan H Y 2001 *Entangled State Representation in Quantum Mechanics and Their Application* (Shanghai: Shanghai Jiaotong University Press) (in Chinese)

[4] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[5] Preskill J 1998 *Quantum Information and Computation* (Singapore: World Scientific)
Nielsen M A and Chuang I L 1995 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[6] Cai X H 2003 *Chin. Phys.* **12** 1066

[7] Fan H Y and Wünsche A 2000 *J. Opt.* B: *Quantum Semiclass. Opt.* **2** 464
Fan H Y and Wünsche A 2003 *Commun. Theor. Phys.* **39** 717

[8] Louisell W H 1973 *Quantum Statistical Properties of Radiation* (New York: Wiley)