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A unified approach in seeking the solitary wave solutions to sine–Gordon type equations*

Xie Yuan-Xi(谢元喜)† and Tang Jia-Shi(唐驾时)

Department of Engineering Mechanics, Hunan University, Changsha 410082, China

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By utilizing the solutions of an auxiliary ordinary differential equation introduced in this paper, we present a simple and direct method to uniformly construct the exact solitary wave solutions for sine–Gordon type equations. As illustrative examples, the exact solitary wave solutions of some physically significant sine–Gordon type equations, including the sine–Gordon equation, double sine–Gordon equation and mKdV–sine–Gordon equation, are investigated by means of this method.

Keywords: auxiliary ordinary differential equation, solitary wave solution, sine–Gordon type equations

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1. Introduction

Since the world around us is inherently nonlinear, nonlinear partial differential equations (NPDEs) are widely used to describe complex phenomena in various fields of sciences, especially in physics. Therefore, exploring the explicit and exact solutions, in particular the solitary wave ones, to NPDEs plays a very important and significant part in the nonlinear sciences. In recent years, a vast variety of powerful and efficient methods have been developed to construct the explicit and exact solutions of NPDEs. Some of the important methods are the function transformation method,[1,2] the homogeneous balance method,[3−8] the tanh-function method,[9,10] the sech-function method,[11,12] the nonlinear transformation method,[13,14] the sine–cosine method,[15,16] the trial function method,[17,18] the superposition method,[19] and so on. Although the above methods are effective, they cannot be applied to directly find the exact solitary wave solutions of sine–Gordon type equations without utilizing some "pre-processing techniques". The present work is motivated by the desire to take full advantage of the abundant types of solutions for an auxiliary ordinary differential equation to put forward a uniform approach for deriving the exact solitary wave solutions to sine–Gordon type equations. The basic idea of our technique is as follows. For a given NPDE, say, in two independent variables x and t

$$N(u, u_t, u_x, u_{xt}, \ldots) = f(u), \tag{1}$$

where the subscripts represent partial derivatives, and $f(u)$ is a known linear combination of $\sin u$ and/or $\sin 2u$. We seek its travelling wave solutions of physical significance, if available, in the following frame

$$u = u(\xi),\ \xi = x - ct. \tag{2}$$

Substituting Eq.(2) into Eq.(1) yields the following ordinary differential equation (ODE) for $u(\xi)$

$$H(u', u'', \ldots) = f(u), \tag{3}$$

where primes stand for the derivatives in terms of its argument.

In order to solve Eq.(3) easily, here we suppose that $u(\xi)$ satisfies the following additional ODE:

$$u' = a_0 + a_1 \cos(ku) + a_2 \sin(ku), \tag{4}$$

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†E-mail: xieyuanxi88@163.com

where a_0, a_1 and a_2 are constants to be determined, and k can be determined by expanding all multiplying terms and equating the highest multiples of u in those sine and cosine functions on both sides in Eq.(3). Substituting Eq.(4) into Eq.(3) and then equating the coefficients of those sine and cosine functions result in a system of algebraic equations, from which the constants a_0, a_1, a_2 and c are determined explicitly. If any of them is left unspecified, then it will be regarded as being arbitrary. By inserting these constants into Eq.(4) and solving it, we can find the solutions to Eq.(1). Fortunately, Eq.(4) admits the following several types of solutions.

Case 1: $a_0 = a_1 = 0, a_2 \neq 0$

$$u(\xi) = \begin{cases} \pm\dfrac{2}{k}\arctan[\exp(ka_2\xi + \xi_0)] \\ \pm\dfrac{2}{k}\arccos\sqrt{\dfrac{1}{2}[1 + \operatorname{sech}(ka_2\xi + \xi_0)]} - \dfrac{\pi}{2k}. \end{cases} \tag{5}$$

Case 2: $a_0 = a_2 = 0, a_1 \neq 0$

$$u(\xi) = \begin{cases} \pm\dfrac{2}{k}\arctan[\exp(ka_1\xi + \xi_0)] - \dfrac{\pi}{2k} \\ \pm\dfrac{2}{k}\arccos\sqrt{\dfrac{1}{2}[1 + \operatorname{sech}(ka_1\xi + \xi_0)]}. \end{cases} \tag{6}$$

Case 3: $a_1 = 0, a_0 \neq 0, a_2 \neq 0$

$$u(\xi) = \begin{cases} \dfrac{2}{k}\arctan\left[\sqrt{\dfrac{a_0+a_2}{a_0-a_2}}\tan\left(\dfrac{k\sqrt{a_0^2-a_2^2}}{2}(\xi+\xi_0)\right)\right] + \dfrac{\pi}{2k} & \text{for } a_0^2 > a_2^2, \\ \dfrac{2}{k}\arctan\left[\sqrt{\dfrac{a_2+a_0}{a_2-a_0}}\coth\left(\dfrac{k\sqrt{a_2^2-a_0^2}}{2}(\xi+\xi_0)\right)\right] + \dfrac{\pi}{2k} & \text{for } a_0^2 < a_2^2 \\ \dfrac{2}{k}\arctan[ka_0(\xi+\xi_0)] + \dfrac{\pi}{2k} & \text{for } a_2 = a_0 \\ -\dfrac{2}{k}\operatorname{arc\,cot}[ka_0(\xi+\xi_0)] + \dfrac{\pi}{2k} & \text{for } a_2 = -a_0. \end{cases} \tag{7}$$

Case 4: $a_2 = 0, a_0 \neq 0, a_1 \neq 0$

$$u(\xi) = \begin{cases} \dfrac{2}{k}\arctan\left[\sqrt{\dfrac{a_0+a_1}{a_0-a_1}}\tan\left(\dfrac{k\sqrt{a_0^2-a_1^2}}{2}(\xi+\xi_0)\right)\right] & \text{for } a_0^2 > a_1^2, \\ \dfrac{2}{k}\arctan\left[\sqrt{\dfrac{a_1+a_0}{a_1-a_0}}\coth\left(\dfrac{k\sqrt{a_1^2-a_0^2}}{2}(\xi+\xi_0)\right)\right] & \text{for } a_0^2 < a_1^2, \\ \dfrac{2}{k}\arctan[ka_0(\xi+\xi_0)] & \text{for } a_1 = a_0, \\ -\dfrac{2}{k}\operatorname{arc\,cot}[ka_0(\xi+\xi_0)] & \text{for } a_1 = -a_0. \end{cases} \tag{8}$$

Case 5: $a_0 = 0, a_1 \neq 0, a_2 \neq 0$

$$u(\xi) = \frac{2}{k}\arctan\left[\frac{\sqrt{a_1^2+a_2^2}}{a_1}\tanh\left(\frac{k\sqrt{a_1^2+a_2^2}}{2}(\xi+\xi_0)\right) + \frac{a_2}{a_1}\right]. \tag{9}$$

In what follows, we shall make use of the approach stated above to find the exact solitary wave solutions of some physically significant sine–Gordon-type equations.

2. Applications of the proposed approach

2.1. The sine–Gordon equation

The well-known sine–Gordon equation under consideration reads

$$u_{xt} = \sin(u). \tag{10}$$

In mathematics it has arisen classically in the study of differential geometry and in physics it has arisen in the study of Josephson junctions, models of particle physics, stability of fluid motions,[20,21] etc. Equation (10) is thought to be completely integrable because it can be solved by inverse scattering transform.[22]

In view of Eq.(2), Eq.(10) can be reduced to the following ODE:

$$-cu'' = \sin(u). \tag{11}$$

Using the ansatz (4) and equating the highest multiples of u in those sine and cosine functions on

both sides of Eq.(11), we obtain $k = \frac{1}{2}$, so that

$$u' = a_0 + a_1 \cos\left(\frac{1}{2}u\right) + a_2 \sin\left(\frac{1}{2}u\right). \tag{12}$$

Inserting Eq.(12) into Eq.(11) and equating the coefficients of those sine and cosine functions leads to a set of algebraic equations with respect to the undetermined constants a_0, a_1, a_2 and c; and solving the resulting system of algebraic equations with the help of Mathematica, we obtain

$$a_0 = 0, \quad a_2 = 0, \quad a_1 \neq 0, \quad c = \frac{4}{a_1^2}; \tag{13}$$

$$a_0 = 0, \quad a_1 = 0, \quad a_2 \neq 0, \quad c = -\frac{4}{a_2^2}. \tag{14}$$

Using Eq.(5) together with Eq.(6), we obtain the following solitary wave solutions for the sine–Gordon equation

$$u(x,t) = \pm 4 \arctan\left[\exp\left(\frac{a_1}{2}x - \frac{2}{a_1}t + \xi_0\right)\right] - \pi, \tag{15}$$

$$u(x,t) = \pm 4 \arccos\sqrt{\frac{1}{2}\left[1 + \operatorname{sech}\left(\frac{a_1}{2}x - \frac{2}{a_1}t + \xi_0\right)\right]}, \tag{16}$$

$$u(x,t) = \pm 4 \arctan\left[\exp\left(\frac{a_2}{2}x + \frac{2}{a_2}t + \xi_0\right)\right], \tag{17}$$

$$u(x,t) = \pm 4 \arccos\sqrt{\frac{1}{2}\left[1 + \operatorname{sech}\left(\frac{a_2}{2}x + \frac{2}{a_2}t + \xi_0\right)\right]} - \pi, \tag{18}$$

where a_1, a_2 and ξ_0 are arbitrary constants.

2.2. The double sine–Gordon equation

The celebrated double sine–Gordon equation in question reads

$$u_{xt} = \sin(u) + \lambda \sin(2u), \tag{19}$$

which is a frequent object of study in numerous physics applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics.[23–25]

Likewise, according to Eq.(2), Eq.(19) can be transformed to the following ODE:

$$-cu'' = \sin(u) + \lambda \sin(2u). \tag{20}$$

Utilizing the ansatz (4) and equating the highest multiples of u in those sine and cosine functions on both sides of Eq.(20), we obtain $k = 1$, and hence

$$u' = a_0 + a_1 \cos(u) + a_2 \sin(u). \tag{21}$$

Substituting Eq.(21) into Eq.(20) and equating the coefficients of those sine and cosine functions yield a system of algebraic equations with regard to the undetermined constants a_0, a_1, a_2 and c, and then solving the resultant system of algebraic equations with the aid of Mathematica, we have

$$a_1 = 2\lambda a_0, \quad a_2 = 0, \quad a_0 \neq 0, \quad c = \frac{1}{2\lambda a_0^2}. \tag{22}$$

Using Eq.(8), we obtain the following solitary wave solutions of the double sine–Gordon equation:

$$u(x,t) = 2 \arctan\left[\sqrt{\frac{1+2\lambda}{1-2\lambda}} \tan\left(\frac{|a_0|\sqrt{1-4\lambda^2}}{2}\left(x - \frac{1}{2\lambda a_0^2}t + \xi_0\right)\right)\right] \quad \text{for} \quad |\lambda| < \frac{1}{2}, \tag{23}$$

$$u(x,t) = 2 \arctan\left[\sqrt{\frac{2\lambda+1}{2\lambda-1}} \coth\left(\frac{|a_0|\sqrt{4\lambda^2-1}}{2}\left(x - \frac{1}{2\lambda a_0^2}t + \xi_0\right)\right)\right] \quad \text{for} \quad |\lambda| > \frac{1}{2}, \tag{24}$$

$$u(x,t) = 2 \arctan\left[a_0\left(x - \frac{1}{a_0^2}t + \xi_0\right)\right] \quad \text{for} \quad \lambda = \frac{1}{2}, \tag{25}$$

$$u(x,t) = -2 \operatorname{arc\,cot}\left[a_0\left(x - \frac{1}{a_0^2}t + \xi_0\right)\right] \quad \text{for} \quad \lambda = -\frac{1}{2}, \tag{26}$$

where a_0 and ξ_0 are arbitrary constants.

2.3. The mKdV–sine–Gordon equation

The renowned mKdV–sine–Gordon equation we are concerned with reads

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sin u, \tag{27}$$

which describes the nonlinear wave propagation in a one-dimensional mono-atomic lattice in which the anharmonic potential competes with the dislocation potential and which can be solved by the inverse scattering transform.[26]

Similarly, based on Eq.(2), Eq.(19) can be changed to the following ODE:

$$-cu'' + \frac{3}{2}(u')^2 u'' + u^{(4)} = \sin(u). \tag{28}$$

Applying the ansatz (4) and equating the highest multiples of u in those sine and cosine functions on both sides of Eq.(28), we obtain $k = \frac{1}{2}$, and hence

$$u' = a_0 + a_1 \cos\left(\frac{1}{2}u\right) + a_2 \sin\left(\frac{1}{2}u\right). \tag{29}$$

Putting Eq.(29) into Eq.(28) and equating the coefficients of those sine and cosine functions give rise to a system of algebraic equations for the undetermined constants a_0, a_1, a_2 and c, and then solving the obtained system of algebraic equations by means of Mathematica, we obtain

$$a_0 = 0, \quad a_2 = 0, \quad a_1 \neq 0, \quad c = \frac{a_1^4 + 16}{4a_1^2}, \tag{30}$$

$$a_0 = 0, \quad a_1 = 0, \quad a_2 \neq 0, \quad c = \frac{a_2^4 - 16}{4a_2^2}. \tag{31}$$

Utilizing Eq.(5) along with Eq.(6), we have the following solitary wave solutions to the mKdV–sine–Gordon equation:

$$u(x,t) = \pm 4 \arctan\left[\exp\left(\frac{a_1}{2}x - \frac{a_1^4 + 16}{8a_1}t + \xi_0\right)\right] - \pi, \tag{32}$$

$$u(x,t) = \pm 4 \arccos\sqrt{\frac{1}{2}\left[1 + \operatorname{sech}\left(\frac{a_1}{2}x - \frac{a_1^4 + 16}{8a_1}t + \xi_0\right)\right]}, \tag{33}$$

$$u(x,t) = \pm 4 \arctan\left[\exp\left(\frac{a_2}{2}x - \frac{a_2^4 - 16}{8a_2}t + \xi_0\right)\right], \tag{34}$$

$$u(x,t) = \pm\, 4 \arccos\sqrt{\frac{1}{2}\left[1 + \operatorname{sech}\left(\frac{a_2}{2}x - \frac{a_2^4 - 16}{8a_2}t + \xi_0\right)\right]} - \pi, \tag{35}$$

where a_1, a_2 and ξ_0 are arbitrary constants.

3. Conclusions

In summary, the exact solitary wave solutions of some physically significant sine–Gordon type equations, including the sine–Gordon equation, double sine–Gordon equation and mKdV–sine–Gordon equation, are presented in a systematic and unified way. The key idea of our approach is to take full advantage of the solutions of an auxiliary ODE introduced in this paper. Its advantage is to transform the problem for solving a NPDE to that for solving an ODE whose solutions are well known. Therefore, the problem under consideration is significantly simplified. How to generalize this technique to solve other NPDEs in mathematical physics is worthy of further study.

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