Vol 14 No 7, July 2005
1009-1963/2005/14(07)/1296-07

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

New expansion algorithm of three Riccati equations and its applications in nonlinear mathematical physics equations

Zhi Hong-Yan(智红燕)[a)†], Zhao Xue-Qin(赵雪芹)[a)b)], and Zhang Hong-Qing(张鸿庆)[a)]

[a)] *Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China*

[b)] *Department of Mathematics, Qufu Normal University, Qufu 273165, China*

(Received 30 December 2004; revised manuscript received 21 March 2005)

Based on the study of tanh function method and the coupled projective Riccati equation method, we propose a new algorithm to search for explicit exact solutions of nonlinear evolution equations. We use the higher-order Schrödinger equation and mKdV equation to illustrate this algorithm. As a result, more new solutions are obtained, which include new solitary solutions, periodic solutions, and singular solutions. Some new solutions are illustrated in figures.

Keywords: symbolic computation, solitary wave solution, periodic solution, higher-order Schrödinger equation, mKdV equation

PACC: 0340K, 0220, 0290

1. Introduction

To find exact solutions of nonlinear evolution equations has long been one of the central subject in mathematics and physics. With the development of soliton theory, many powerful methods have been presented, such as the inverse scattering transform method,[1,2] Bäcklund transformation,[2,3] Hirota's bilinear method,[4] homogeneous balance method,[5] various tanh methods,[6−15] the projective Riccati equation method[16] and the similarity reduction method,[17] etc.

It is known that the Riccati equation $\phi' = R + \phi^2$ has the following solutions:

$$\phi_1 = -\frac{\sqrt{-R}\sinh(\sqrt{-R}\xi)}{\cosh(\sqrt{-R}\xi)}, \qquad R < 0,$$

$$\phi_2 = \frac{\sqrt{R}\sin(\sqrt{R}\xi)}{\cos(\sqrt{R}\xi)}, \qquad R < 0,$$

and that the coupled projective Riccati equations $\sigma' = \epsilon\sigma\tau$ and $\tau' = R + \epsilon\tau^2 - \mu\sigma$ have the solutions:

$$\sigma = \frac{R}{\mu + \cosh(\sqrt{R}\xi)},$$

$$\tau = \frac{\sqrt{R}\sinh(\sqrt{R}\xi)}{\mu + \cosh(\sqrt{R}\xi)},$$

$$\epsilon = -1, R \neq 0,$$

or

$$\sigma = \frac{R}{\mu + \cos(\sqrt{R}\xi)},$$

$$\tau = \frac{\sqrt{R}\sin(\sqrt{R}\xi)}{\mu + \cos(\sqrt{R}\xi)},$$

$$\epsilon = 1, R \neq 0.$$

Inspired by the above knowledge, we wonder whether there is a system that possesses the following solutions:

$$f = \frac{\sinh(\xi)}{1 + A\cosh^2(\xi)},$$

$$g = \frac{\cosh(\xi)}{1 + A\cosh^2(\xi)},$$

$$h = \frac{1}{1 + A\cosh^2(\xi)},$$

or

$$f = \frac{\sin(\xi)}{1 + A\cos^2(\xi)},$$

$$g = \frac{\cos(\xi)}{1 + A\cos^2(\xi)},$$

†E-mail: zhihongyan@126.com

$$h=\frac{1}{1+A\cos^2(\xi)}.$$

Through calculation, we create a system of three Riccati equations.

In this paper, we introduce a new method and apply it to the following higher-order Schrödinger equation:[18]

$$\begin{aligned}\frac{\partial\Psi}{\partial z}=&\mathrm{i}\alpha_1\frac{\partial^2\Psi}{\partial t^2}+\mathrm{i}\alpha_2\Psi|\Psi|^2+\alpha_3\frac{\partial^3\Psi}{\partial t^3}\\&+\alpha_4\frac{\partial\Psi|\Psi|^2}{\partial t}+\alpha_5\Psi\frac{\partial|\Psi|^2}{\partial t}\end{aligned}\tag{1}$$

and mKdV equation

$$u_t+6u^2u_x+u_{xxx}=0.\tag{2}$$

In Eq.(1), Ψ is a slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivative in retard time coordinates, and α_1, α_2, α_3, α_4 and α_5 are the real parameters related to the group velocity, self-phase modulation, third-order dispersion, self-steepening, and self-frequency shift arising from stimulated Raman scattering, respectively. Some properties of the equation, as well as many versions of it, have been studied.[18–25] Up to now, the bright, dark and the combined bright and dark solitary waves and periodic waves are found from Eq.(1) and its special case. In this paper, we apply our method to Eq.(1) so that new exact solutions are obtained.

The rest of this paper is arranged as follows. In Section 2, we briefly describe our method. In Section 3, we apply it to the Schrödinger equation (1) and mKdV equation (2) to illustrate our algorithm. Finally, some conclusions are given in Section 4.

2. New expansion algorithm of three Riccati equations

In this section, we describe the method step by step as follows. For a given partial differential equation for $u(x,t)$:

$$F(u,u_t,u_x,u_{xx},\ldots)=0.\tag{3}$$

Under the transformation $u(x,t)=u(\xi)$ and $\xi=x+\lambda t$, where λ is a constant to be determined, Eq.(3) reduces to

$$G(u,u',u'',u'''\ldots)=0,\tag{4}$$

where $'=\frac{\mathrm{d}}{\mathrm{d}\xi}$. We seek the solutions of Eq.(4) in the form

$$u=a_0+\sum_{i=1}^{n}h^{i-1}(a_if+b_ig+c_ih),\tag{5}$$

where f,g,h satisfy

$$\begin{aligned}f'&=k\delta g(1-2h-2\epsilon Ah),\\g'&=\delta f(1-2h),\\h'&=2\delta Afg,\\Af^2&=kh(1-h-\epsilon Ah),\\Ag^2&=h(1-h),\end{aligned}\tag{6}$$

and a_i, b_i, c_i, $A\neq0$, n are constants to be determined later.

It is easy to see that Eq.(6) has the following solutions:
when $k=1,\delta=-1,\epsilon=1$

$$\begin{aligned}f(\xi)&=\frac{\sinh(\xi)}{1+A\cosh^2(\xi)},\\g(\xi)&=\frac{\cosh(\xi)}{1+A\cosh^2(\xi)},\\h(\xi)&=\frac{1}{1+A\cosh^2(\xi)},\end{aligned}\tag{7}$$

when $k=1,\delta=-1,\epsilon=-1$

$$\begin{aligned}f(\xi)&=\frac{\cosh(\xi)}{1+A\sinh^2(\xi)},\\g(\xi)&=\frac{\sinh(\xi)}{1+A\sinh^2(\xi)},\\h(\xi)&=\frac{1}{1+A\sinh^2(\xi)},\end{aligned}\tag{8}$$

when $k=-1,\epsilon=1,\delta=1$

$$\begin{aligned}f(\xi)&=\frac{\sin(\xi)}{1+A\cos^2(\xi)},\\g(\xi)&=\frac{\cos(\xi)}{1+A\cos^2(\xi)},\\h(\xi)&=\frac{1}{1+A\cos^2(\xi)},\end{aligned}\tag{9}$$

when $k=-1,\epsilon=1,\delta=-1$

$$\begin{aligned}f(\xi)&=\frac{\cos(\xi)}{1+A\sin^2(\xi)},\\g(\xi)&=\frac{\sin(\xi)}{1+A\sin^2(\xi)},\\h(\xi)&=\frac{1}{1+A\sin^2(\xi)}.\end{aligned}\tag{10}$$

To determine u explicitly, we take the following steps:
step 1 Determine n by balancing the highest-order partial differential term and the nonlinear terms in Eq.(3) or Eq.(4).
step 2 With the aid of Maple, substitute Eq.(5)

and Eq.(6) into Eq.(4) and then collect the coefficients of the same power $f^i g^j h^l (i = 0, 1, j = 0, 1, l = 0, 1, 2, 3...)$. Set each of the obtained coefficients to zero to obtain an over-determined system of algebraic equations for $\lambda, a_i, b_i, c_i, A$.

step 3 Solve the system of algebraic equations obtained in step 2 for $\lambda, a_i, b_i, c_i, A$.

step 4 Substitute these values into Eqs.(5) along with Eqs.(7), (8), (9) and (10) to obtain the solutions of Eq.(3).

Remark Apparently, the solutions of Eqs.(6) are different from those of Riccati equation and projective Riccati equations.

3. Exact solutions of the equations

Example 1

In order to seek the exact solutions of Eq.(1), according to the above steps, we make the gauge transformation:

$$\Psi(z,t) = \psi(\xi)\exp[\mathrm{i}(kz - wt], \quad \xi = t - \lambda z, \tag{11}$$

where λ, w, k are constants to be determined later. Substituting Eq.(11) into Eq.(1) yields a complex ordinary differential equation for ψ, the real and imaginary parts of which respectively read as

$$(\alpha_1 - 3\alpha_3 w)\psi'' + (\alpha_3 w^3 - \alpha_1 w^2)\psi + (\alpha_2 - \alpha_4 w)\psi^3 = 0, \tag{12a}$$

$$\alpha_3\psi''' + (2\alpha_1 w - 3\alpha_3 w^2 + \lambda)\psi' + (3\alpha_4 + 2\alpha_5)\psi^2\psi' = 0. \tag{12b}$$

It is easy to see that Eqs.(12a) and (12b) are combined into an equation

$$\psi'' + \frac{2\alpha_1 w - 3\alpha_3 w^2 + \lambda}{\alpha_3}\psi + \frac{3\alpha_4 + 2\alpha_5}{3\alpha_3}\psi^3 = 0, \tag{13}$$

under the constraint condition

$$w = \frac{\alpha_1(3\alpha_4 + 2\alpha_5) - 3\alpha_2\alpha_3}{6\alpha_3(\alpha_4 + \alpha_5)},$$

$$k = -\frac{1}{\alpha_3}[(\alpha_1 - 3\alpha_3 w) \times (2\alpha_1 w - 3\alpha_3 w^2 + \lambda) - \alpha_1 w^2] + w^3.$$

By balancing the highest order derivative and nonlinear terms in Eq.(13), we obtain $n = 1$. In what follows, we will obtain the solutions of Eq.(1) by considering Eq.(13). According to the method, we assume that Eq.(13) has the following solution:

$$\psi = a_0 + a_1 f + a_2 g + a_3 h, \tag{14}$$

where a_0, a_1, a_2, a_3 are constants to be determined later.

With the aid of Maple 9, substitute Eq.(14) and Eq.(6) into Eq.(13), and collect all terms with the same power in $f^i g^j h^l (i = 0, 1, j = 0, 1, l = 0, 1, 2, 3, 4)$. Setting the coefficients of these terms $f^i g^j h^l$ to zero yields a set of algebraic polynomial equations. For simplicity, we omit them. With the aid of Maple 9, we have the following constants of a_0, a_1, a_2, a_3, A, and λ.

When $k = 1, \delta = -1$, and $\epsilon = 1$

Case 1

$a_3 = 2\sqrt{\dfrac{6\alpha_3}{3\,\alpha_4 + 2\,\alpha_5}}, a_2 = a_0 = a_1 = 0, A = -2$, and $\lambda = -4\,\alpha_3 - 2\,\alpha_1 w + 3\,\alpha_3 w^2$;

Case 2

$a_1 = -\sqrt{\dfrac{6\alpha_3 A}{3\,\alpha_4 + 2\,\alpha_5}}$, $a_2 = \sqrt{\dfrac{6\,\alpha_3 A + 6\,\alpha_3 A^2}{3\,\alpha_4 + 2\,\alpha_5}}$, $a_0 = a_3 = 0$, and $\lambda = -2\,\alpha_1 w - \alpha_3 + 3\,\alpha_3 w^2$,

where A is an arbitrary constant.

When $k = 1, \delta = -1$, and $\epsilon = -1$

Case 3

$a_0 = a_3 = 0, \lambda = -2\,\alpha_1 w - \alpha_3 + 3\,\alpha_3 w^2, a_2 = \sqrt{-\dfrac{-6\,A\alpha_3 + 6\,\alpha_3 A^2}{3\,\alpha_4 + 2\,\alpha_5}}$, and $a_1 = \sqrt{\dfrac{6\,A\alpha_3}{3\,\alpha_4 + 2\,\alpha_5}}$,

where A is an arbitrary constant;

Case 4

$A = 2, a_0 = a_1 = a_2 = 0, \lambda = -4\,\alpha_3 + 3\,\alpha_3 w^2 - 2\,\alpha_1 w$, and $a_3 = 2\sqrt{\dfrac{6\,\alpha_3}{3\,\alpha_4 + 2\,\alpha_5}}$.

When $k = -1, \epsilon = 1$, and $\delta = 1$

Case 5

$a_0 = a_2 = a_1 = 0, \lambda = -2\,\alpha_1 w + 3\,\alpha_3 w^2 + 4\,\alpha_3, A = -2$, and $a_3 = 2\sqrt{-\dfrac{6\alpha_3}{3\,\alpha_4 + 2\,\alpha_5}}$;

Case 6

$a_0 = a_3 = 0, a_1 = \sqrt{\dfrac{6\alpha_3 A}{3\,\alpha_4 + 2\,\alpha_5}}, \lambda = 3\,\alpha_3 w^2 - 2\,\alpha_1 w + \alpha_3$, and $a_2 = \sqrt{-\dfrac{6\,\alpha_3 A + 6\,\alpha_3 A^2}{3\,\alpha_4 + 2\,\alpha_5}}$,

where A is an arbitrary constant.

When $k = -1, \epsilon = 1$, and $\delta = -1$

Case 7

$a_1 = a_0 = a_2 = 0, A = -2, \lambda = -2\,\alpha_1 w + 4\,\alpha_3 + 3\,\alpha_3 w^2$, and $a_3 = 2\sqrt{-\dfrac{6\alpha_3}{3\,\alpha_4 + 2\,\alpha_5}}$;

Case 8

$a_0 = a_3 = 0, a_1 = \sqrt{\dfrac{6\alpha_3 A}{3\,\alpha_4 + 2\,\alpha_5}}, \lambda = -2\,\alpha_1 w + 3\,\alpha_3 w^2 + \alpha_3$, and $a_2 = \sqrt{-\dfrac{6\,\alpha_3 A + 6\,\alpha_3 A^2}{3\,\alpha_4 + 2\,\alpha_5}}$,

where A is an arbitrary constant.

Therefore, we can obtain the following exact solutions:

$$\Psi_1 = \frac{2\sqrt{\dfrac{6\alpha_3}{3\,\alpha_4+2\,\alpha_5}}}{1-2\,\cosh^2(t-\lambda z)}\exp[\mathrm{i}(kz-wt],$$

where $\lambda = -4\alpha_3 - 2\alpha_1 w + 3\alpha_3 w^2$;

$$\Psi_2 = \Bigg\{ -\sqrt{\frac{6\,\alpha_3 A}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\sinh(t-\lambda z)}{1+A\cosh^2(t-\lambda z)} + \sqrt{\frac{6\,\alpha_3 A+6\,\alpha_3 A^2}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\cosh(t-\lambda z)}{1+A\cosh^2(t-\lambda z)} \Bigg\} \times \exp[\mathrm{i}(kz-wt],$$

where $\lambda = -\alpha_3 - 2\alpha_1 w + 3\alpha_3 w^2$;

$$\Psi_3 = \Bigg\{ \sqrt{\frac{6\,A\alpha_3}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\cosh(t-\lambda z)}{1+A\sinh^2(t-\lambda z)} + \sqrt{-\frac{-6\,A\alpha_3+6\,\alpha_3 A^2}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\sinh(t-\lambda z)}{1+A\sinh^2(t-\lambda z)} \Bigg\} \times \exp[\mathrm{i}(kz-wt],$$

where $\lambda = -2\alpha_1 w - \alpha_3 + 3\alpha_3 w^2$;

$$\Psi_4 = \frac{2\sqrt{\dfrac{6\,\alpha_3}{3\,\alpha_4+2\,\alpha_5}}}{1+2\,\sinh^2(t-\lambda z)}\exp[\mathrm{i}(kz-wt],$$

where $\lambda = -4\alpha_3 + 3\alpha_3 w^2 - 2\alpha_1 w$;

$$\Psi_5 = \frac{-2\sqrt{-6\,\dfrac{\alpha_3}{3\,\alpha_4+2\,\alpha_5}}}{1-2\,\cos^2(t-\lambda z)}\exp[\mathrm{i}(kz-wt],$$

where $\lambda = -2\,\alpha_1 w + 3\,\alpha_3 w^2 + 4\,\alpha_3$;

$$\Psi_6 = \Bigg\{ \sqrt{\frac{6\,A\alpha_3}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\sin(t-\lambda z)}{1+A\cos^2(t-\lambda z)} - \sqrt{-\frac{6\,\alpha_3 A^2+6\,A\alpha_3}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\cos(t-\lambda z)}{1+A\cos^2(t-\lambda z)} \Bigg\} \times \exp[\mathrm{i}(kz-wt],$$

where $\lambda = 3\,\alpha_3 w^2 - 2\,\alpha_1 w + \alpha_3$;

$$\Psi_7 = \frac{2\sqrt{-\dfrac{6\alpha_3}{3\,\alpha_4+2\,\alpha_5}}}{1-2\,\sin^2(t-\lambda z)}\exp[\mathrm{i}(kz-wt],$$

where $\lambda = -2\,\alpha_1 w + 4\,\alpha_3 + 3\,\alpha_3 w^2$;

$$\Psi_8 = \Bigg\{ \sqrt{\frac{6\,\alpha_3 A}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\cos(t-\lambda z)}{1+A\sin^2(t-\lambda z)} + \sqrt{-\frac{6\,\alpha_3 A+6\,\alpha_3 A^2}{3\,\alpha_4+2\,\alpha_5}} \times \frac{\sin(t-\lambda z)}{1+A\sin^2(t-\lambda z)} \Bigg\} \times \exp[\mathrm{i}(kz-wt],$$

where $\lambda = -2\,\alpha_1 w + 3\,\alpha_3 w^2 + \alpha_3$.

Remark Among the above solutions, Ψ_2 and Ψ_3 are both new forms of exact solutions. And they are illustrated in Fig.1 and Fig.2, respectively.

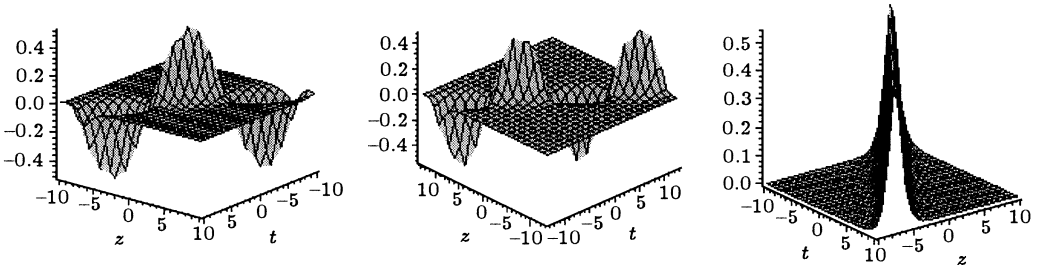


Fig.1. Solitary solution of Schrödinger equation Ψ_2, the real part, imaginary part and the modulus, where $A = 14, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 5$, and $\alpha_5 = 3$.

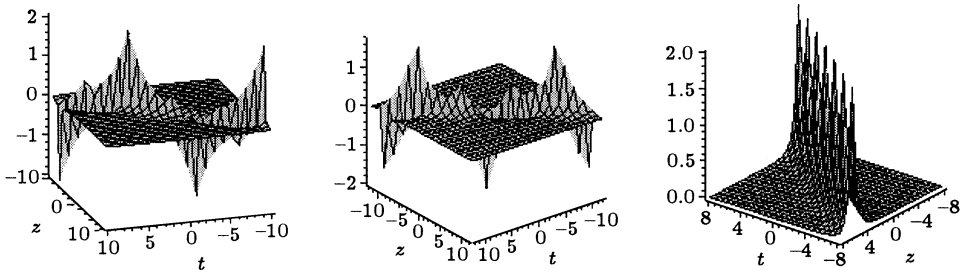


Fig.2. Solitary solution of Schrödinger equation Ψ_3, the real part, imaginary part and the modulus, where $a_1 = 10$ and $a_2 = 0.1$.

Example 2

At first, we make the travelling transformation

$$u = u(\xi), \qquad \xi = kx + \lambda t, \tag{15}$$

and Eq.(2) becomes

$$\lambda u' + 6ku^2u' + k^3u''' = 0. \tag{16}$$

Therefore we obtain the following solutions:

$$u_1 = a_1 \operatorname{csch}(\xi), \qquad \xi = -\mathrm{i}a_1 x - \mathrm{i}{a_1}^3 t,$$

where a_1 is an arbitrary constant;

$$u_2 = \frac{-{a_3}^2 + 4\,k^2}{4a_3} + \frac{4k^2 a_3}{4k^2 - (4\,k^2 + {a_3}^2)\cosh^2(\xi)},$$

$$\xi = kx - \frac{k\left(8\,k^2{a_3}^2 + 3{a_3}^4 + 48k^4\right)t}{8{a_3}^2},$$

where k, a_3 are arbitrary constants;

$$u_3 = \frac{{a_3}^3}{2({a_2}^2 - {a_3}^2)} + \frac{a_2 a_3^2 \cosh(\xi)}{a_3^2 - {a_2}^2\cosh^2(\xi)} + \frac{a_3^3}{a_3^2 - {a_2}^2\cosh^2(\xi)},$$

$$\xi = \frac{{a_3}^2 x}{\sqrt{{a_2}^2 - {a_3}^2}} - \frac{{a_3}^6\left(2\,{a_2}^2 + {a_3}^2\right)t}{2\sqrt{({a_2}^2 - {a_3}^2)}\left({a_2}^2 - {a_3}^2\right)^2},$$

where a_2, a_3 are arbitrary constants;

$$u_4 = -\frac{a_3}{2} + \frac{a_1(a_1^2 + a_3^2)\sinh(\xi)}{a_1^2 + a_3^2 - {a_1}^2\cosh^2(\xi)} + \frac{a_3(a_1^2 + a_3^2)}{a_1^2 + a_3^2 - {a_1}^2\cosh^2(\xi)},$$

$$\xi = -\sqrt{-{a_1}^2 - {a_3}^2}\left(x + \frac{\left(2\,{a_1}^2 - {a_3}^2\right)t}{2}\right),$$

where a_1, a_3 are arbitrary constants;

$$u_5 = \frac{a_1^3\sinh(\xi)}{a_1^2 - ({a_1}^2 - {a_2}^2)\cosh^2(\xi)} + \frac{{a_1}^2 a_2\cosh(\xi)}{{a_1}^2 - ({a_1}^2 - {a_2}^2)\cosh^2(\xi)},$$

$$\xi = \frac{{a_1}^2 x}{\sqrt{({a_2}^2 - {a_1}^2)}} + \frac{{a_1}^6 t}{\sqrt{({a_2}^2 - {a_1}^2)}({a_1}^2 - {a_2}^2)},$$

where a_1, a_2 are arbitrary constants;

$$u_6 = -k\operatorname{sech}(\xi), \qquad \xi = kx - k^3 t,$$

where k is an arbitrary constant;

$$u_7 = \frac{\mathrm{i}\sqrt{2}k}{4} - \frac{\mathrm{i}\sqrt{2}k\sinh(\xi)}{1 - \sinh^2(\xi)} - \frac{\mathrm{i}\sqrt{2}k}{1 - \sinh^2(\xi)},$$

$$\xi = kx - \frac{k^3 t}{4},$$

where k is an arbitrary constant;

$$u_8 = \frac{\sqrt{A}k\cosh(\xi)}{1 + A\sinh^2(\xi)} + \frac{\sqrt{-A^2 + A}k\sinh(\xi)}{1 + A\sinh^2(\xi)},$$

$$\xi = kx - k^3 t,$$

where k, A are arbitrary constants;

$$u_9 = \frac{k^2}{2a_3} + \frac{k\sqrt{-k^2 - {a_3}^2}a_3\sinh(\xi)}{k^2 + (k^2 + {a_3}^2)\sinh^2(\xi)} + \frac{a_3 k^2}{k^2 + (k^2 + {a_3}^2)\sinh^2(\xi)},$$

$$\xi = kx - \frac{k^3\left(2\,{a_3}^2 + 3\,k^2\right)t}{2{a_3}^2},$$

where k and a_3 are arbitrary constants;

$$u_{10} = \frac{4\,k^2 - {a_3}^2}{4a_3} + \frac{4k^2 a_3}{4k^2 + (4\,k^2 + {a_3}^2)\sinh^2(\xi)},$$

$$\xi = kx - \frac{k\left(48\,k^4 + 8\,k^2{a_3}^2 + 3\,{a_3}^4\right)t}{8{a_3}^2},$$

where k and a_3 are arbitrary constants;

$$u_{11} = -\frac{a_3}{2} + \frac{k^2\sqrt{k^2 + {a_3}^2}\cosh(\xi)}{k^2 + (k^2 + {a_3}^2)\sinh^2(\xi)} + \frac{a_3 k^2}{k^2 + (k^2 + {a_3}^2)\sinh^2(\xi)},$$

$$\xi = kx - \left(k^3 + \frac{3}{2}k{a_3}^2\right)t,$$

where k and a_3 are arbitrary constants;

$$u_{12} = -\frac{{a_3}^3}{2({a_3}^2 + {a_2}^2)} + \frac{a_2{a_3}^2\sinh(\xi)}{{a_3}^2 - {a_2}^2\sinh^2(\xi)} + \frac{{a_3}^3}{{a_3}^2 - {a_2}^2\sinh^2(\xi)},$$

$$\xi = -\frac{{a_3}^2 x}{\sqrt{-({a_3}^2 + {a_2}^2)}} + \frac{{a_3}^6\left(-2\,{a_2}^2 + {a_3}^2\right)t}{2\sqrt{-({a_3}^2 + {a_2}^2)}({a_3}^2 + {a_2}^2)^2},$$

where a_2 and a_3 are arbitrary constants;

$$u_{13} = a_1\csc(\xi), \quad \xi = ia_1 x - \mathrm{i}{a_1}^3 t,$$

where a_1 is an arbitrary constant;

$$u_{14} = \frac{a_1^3\sin(\xi)}{a_1^2 - ({a_1}^2 + {a_2}^2)\cos^2(\xi)} + \frac{a_1^2 a_2\cos(\xi)}{a_1^2 - ({a_1}^2 + {a_2}^2)\cos^2(\xi)},$$

$$\xi = \frac{{a_1}^2 x}{\sqrt{-(a_1^2+a_2^2)}} - \frac{{a_1}^6 t}{\sqrt{-(a_1{}^2+a_2^2)}\,({a_1}^2+a_2^2)},$$

where a_1 and a_2 are arbitrary constants;

$$u_{15} = -\frac{a_3}{2} + \frac{a_1(a_1^2-a_3^2)\sin(\xi)}{{a_1}^2-{a_3}^2-{a_1}^2\cos^2(\xi)} + \frac{a_3({a_1}^2-{a_3}^2)}{{a_1}^2-{a_3}^2-{a_1}^2\cos^2(\xi)},$$

$$\xi = \sqrt{{a_3}^2-{a_1}^2}\left(x - \frac{\left(2\,{a_1}^2+{a_3}^2\right)t}{2}\right),$$

where a_1 and a_3 are arbitrary constants;

$$u_{16} = -\frac{4\,k^2+{a_3}^2}{4a_3} + \frac{4k^2a_3}{4k^2-(4\,k^2-{a_3}^2)\cos^2(\xi)},$$

$$\xi = kx - \frac{k\left(-8\,k^2{a_3}^2+48\,k^4+3\,{a_3}^4\right)t}{8{a_3}^2},$$

where k and a_3 are arbitrary constants;

$$u_{17} = a_1\sec(\xi), \qquad \xi = \mathrm{i}a_1x - \mathrm{i}{a_1}^3t,$$

where a_1 is an arbitrary constants;

$$u_{18} = -\frac{a_3}{4} + \frac{\mathrm{i}a_3\sin(\xi)}{1+\sin^2(\xi)} + \frac{a_3}{1+\sin^2(\xi)},$$

$$\xi = \frac{\sqrt{2}}{2}a_3x + \frac{\sqrt{2}}{16}{a_3}^3t,$$

where a_3 is an arbitrary constant;

$$u_{19} = -\frac{4\,k^2+{a_3}^2}{4a_3} + \frac{4k^2a_3}{4k^2-(4\,k^2-{a_3}^2)\sin^2(\xi)},$$

$$\xi = kx - \frac{k\left(-8\,k^2{a_3}^2+48\,k^4+3\,{a_3}^4\right)t}{8{a_3}^2},$$

where k and a_3 are arbitrary constants;

$$u_{20} = \frac{a_1^3\cos(\xi)}{a_1^2-({a_2}^2+{a_1}^2)\sin^2(\xi)} + \frac{a_1^2a_2\sin(\xi)}{a_1^2-({a_2}^2+{a_1}^2)\sin^2(\xi)},$$

$$\xi = \frac{{a_1}^2x}{\sqrt{-({a_2}^2+{a_1}^2)}} - \frac{{a_1}^6t}{\sqrt{-({a_2}^2+{a_1}^2)}\,({a_2}^2+{a_1}^2)},$$

where a_1 and a_2 are arbitrary constants;

$$u_{21} = \frac{{a_3}^3}{2({a_2}^2-{a_3}^2)} + \frac{a_2a_3^2\sin(\xi)}{a_3^2-{a_2}^2\sin^2(\xi)} + \frac{a_3^3}{{a_3}^2-{a_2}^2\sin^2(\xi)},$$

$$\xi = \frac{-{a_3}^2x}{\sqrt{({a_3}^2-{a_2}^2)}} + \frac{{a_3}^6t}{2\sqrt{({a_3}^2-{a_2}^2)}\,(2\,{a_2}^2+{a_3}^2)\,({a_2}^2-{a_3}^2)^2},$$

where a_2 and a_3 are arbitrary constants;

$$u_{22} = -\frac{a_3}{2} + \frac{a_1\left(a_1^2-a_3^2\right)\cos(\xi)}{(a_1^2-a_3^2)-{a_1}^2\sin^2(\xi)} + \frac{a_3\left(a_1^2-a_3^2\right)}{(a_1^2-a_3^2)-{a_1}^2\sin^2(\xi)},$$

$$\xi = \sqrt{{a_3}^2-{a_1}^2}\left(x - \frac{\left({a_3}^2+2\,{a_1}^2\right)t}{2}\right),$$

where a_1 and a_3 are arbitrary constants.

Remark As far as we know, the solutions $u_3 \sim u_5, u_8 \sim u_{10}, u_{14} \sim u_{16}, u_{20} \sim u_{21}$, are all new forms of solutions. And we use Fig.3 and Fig.4 to describe the solutions u_5 and u_{12}, respectively.

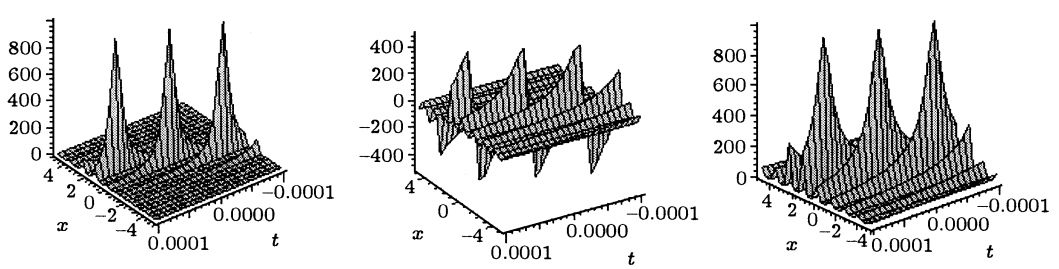


Fig.3. Solitary solution of mKdV equation u_5 the real part, imaginary part and the modulus of u_5, where $A = 14$, $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 5$, and $\alpha_5 = 3$.

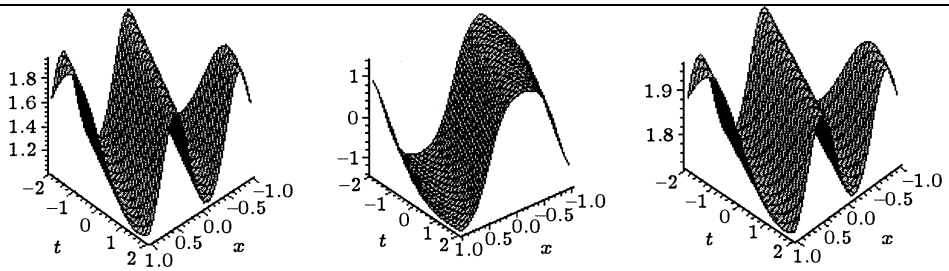


Fig.4. Solitary solution of mKdV equation u_{12}, the real part, imaginary part and the modulus, where $a_2 = 2$ and $a_3 = 3$.

4. Conclusion

In summary, we have derived many families of exact travelling wave solutions of the higher-order nonlinear Schrödinger equation in optical fibres and mKdV equation by using our new method. It is shown that symbolic computation method is a powerful tool for seeking the solutions of nonlinear evolution equations. Our method can be also used to search for the new solution of other nonlinear evolution equations.

References

[1] Gardner C S *et al* 1967 *Phys. Rev. Lett.* **19** 1095

[2] Ablowitz M J and Clarkson P A 1991 *Soliton, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)

[3] Miura M R 1978 *Bäcklund Transformation* (Berlin: Springer)

[4] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192

[5] Wang M L, Zhou Y B and Li Z B 1996 *Phys. Lett.* A **216** 67

[6] Malfliet W 1992 *Am. J. Phys. Scr.* **60** 650
Malfliet W and Hereman W 1996 *Phys. Scr.* **54** 563

[7] Parkes E J and Duffy B R 1996 *Comp. Phys. Commun.* **98** 288
Parkes E J and Duffy B R 1997 *Phys. Lett.* A **229** 217

[8] Fan E G 2000 *Phys.* A **277** 212
Fan E G 2000 *Phys.* A **294** 26

[9] Elwakil S A, El-labany S K, Zahran M A and Sabry R 2002 *Phys. Lett.* A **299** 179

[10] Yan Z Y 2001 *Phys. Lett.* A **285** 355
Yan Z Y 2001 *Phys. Lett.* A **292** 100

[11] Lü Z S and Zhang H Q 2003 *Phys. Lett.* A **307** 269

[12] Li B, Chen Y and Zhang H Q 2003 *Z. Natur.* A **57** 874

[13] Gao Y T and Tian B 2001 *Comp. Phys. Commun.* **133** 158
Tian B and Gao Y T 2002 *Z. Natur.* A **57** 39

[14] Li B, Chen Y, Xuan H and Zhang H Q 2003 *Chaos, Solitons Fractals* **17** 885

[15] Chen Y and Wang Q 2004 *Int. J. Mod. Phys.* C **15** 595

[16] Conte R and Musette M 1992 *J. Phys.* A **25** 5609

[17] Lou S Y 1995 *Math. Method Appl. Sci.* **18** 789

[18] Kodama Y 1985 *J. Stat. Phys.* **39** 597

[19] Gedalin M, Scott T C and Band Y B 1997 *Phys. Rev. Lett.* **78** 448

[20] Palacios S L *et al* 1999 *Phys. Rev.* E **60** 45

[21] Li Z H *et al* 2000 *Phys. Rev. Lett.* **84** 4096

[22] Cheng L 1996 *J. Phys.* A **29** L141

[23] Hizanidis K *et al* 1996 *J. Phys.* A **29** 7687

[24] Potasek M J and Tabor T 1991 *Phys. Lett.* A **154** 449

[25] Yan Z Y 2003 *Chaos, Solitons Fractals* **16** 759