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# Hyperbolic function method for solving nonlinear differential-different equations*

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An algorithm is devised to obtained exact travelling wave solutions of differential-different equations by means of hyperbolic function. For illustration, we apply the method to solve the discrete nonlinear (2+1)-dimensional Toda lattice equation and the discretized nonlinear mKdV lattice equation, and successfully constructed some explicit and exact travelling wave solutions.

**Keywords:** discrete (2+1)-dimensional Toda lattice equation, discretized mKdV lattice equation, travelling wave solutions, hyperbolic function approach

**PACC:** 0340, 0365, 0545

## 1. Introduction

Seeking exact solutions of nonlinear evolution equations is of importance in mathematical physics. In the past decades, several powerful methods have been developed to construct many types of exact solutions of nonlinear partial differential equations(PDEs)−− such as inverse scattering theory,[1] Bäcklund transformation,[2] hyperbolic function method,[3−5] homogeneous balance method,[6−8] multilinear variable separation approach[9−11] and Jacobian elliptic function method[12−15] etc. In recent years, the direct search for exact solutions of PDEs becomes more and more attractive partly due to the availability of computer symbolic systems like Maple or Mathematica, which allows us to perform complicated and tedious algebraic calculations on computer. In particular, one of the most effective direct methods to construct exact solutions of PDEs is the hyperbolic function method.

Since the work of Fermi, Pasta and Ulam,[16] the investigation of exact solutions of the nonlinear differential-different equations (DDEs) plays a crucial role in the modelling of many phenomena in different fields, including condensed matter, biophysics and mechanical engineering. One also encounters such systems in numerical simulation of soliton dynamics in high energy physics where they arise as approximations of continuum models. Unlike difference equations which are fully discretized, differential-different equations are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous. However, different from the considerably many works done on finding exact solutions to PDEs mentioned above, to our knowledge, few work has been done to investigate exact solutions for DDEs. For example, Qian *et al*[17] have successfully extended the multilinear variable separation approach to a special differential-different equation. More recently, Baldwin and Hereman[18] presented an algorithm to find exact travelling wave solutions of DDEs in terms of tanh function and have found kink-type solutions in many spatially discrete nonlinear models such as the celebrated Toda lattice[19] and Ablowitz–Ladik lattice,[20] etc. Dai *et al*[21] extended the approach to the Jacobian elliptic function expansion method and found Jacobian doubly periodic solution and kink-type tanh solitary solution.

In this paper, we develop a new algorithm to obtain some exact travelling solutions of DDEs. For illustration, we apply this method to discrete nonlinear (2+1)-dimensional Toda lattice equation and the

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discrete nonlinear mKdV lattice equation, and derive kink-type solitary solution, bell-type solitary solution and some other hyperbolic function solutions by using the hyperbolic function method. In this sense, our method is superior to the tanh method[18] and different from the Jacobian elliptic function expansion method.[21]

## 2. The hyperbolic function method for nonlinear DDEs

First, we recall the properties of hyperbolic functions,

$$\cosh^2(\xi)-\sinh^2(\xi)=1,\quad \sinh(\xi)/\cosh(\xi)=\tanh(\xi),$$
$$(\sinh(\xi))'=\cosh(\xi),\ (\cosh(\xi))'=\sinh(\xi), \tag{1}$$
$$\sinh(\xi_1\pm\xi_2)=\sinh(\xi_1)\cosh(\xi_2)\pm\cosh(\xi_1)\sinh(\xi_2),$$
$$\cosh(\xi_1\pm\xi_2)=\cosh(\xi_1)\cosh(\xi_2)\pm\sinh(\xi_1)\sinh(\xi_2). \tag{2}$$

Given a system of $M$ polynomial DDEs,

$$\Delta(u_{n+p_1}(x),\cdots,u_{n+p_k}(x),u'_{n+p_1}(x),\cdots,u'_{n+p_k}(x),\cdots,u^{(r)}_{n+p_1}(x),\cdots,u^{(r)}_{n+p_k}(x))=0, \tag{3}$$

where the dependent variable $u$ has $M$ components $u_i$, the continuous variable $x$ has $N$ components $x_i$, the discrete variable $n$ has $Q$ components $n_j$, the $k$ shift vectors $p_i$, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order $r$.

The main steps of the hyperbolic function method are outlined as follows:

**Step 1** When we seek the travelling wave solutions of Eq.(3), the first step is to introduce the wave transformation

$$\xi_n=\sum_{i=1}^{Q}d_in_i+\sum_{j=1}^{N}c_jx_j+\delta, \tag{4}$$

where the coefficients $d_i, c_j$ and the phase $\delta$ are constants to be determined. In this way, Eq.(3) becomes

$$\Delta(u_{n+p_1}(\xi_n),\cdots,u_{n+p_k}(\xi_n),u'_{n+p_1}(\xi),\cdots,u'_{n+p_k}(\xi),u^{(r)}_{n+p_1}(x),\cdots,u^{(r)}_{n+p_k}(x))=0. \tag{5}$$

**Step 2** We propose the following series expansion as a solution of Eq.(3) or (5)

$$u_n\equiv u_n(\xi_n)=\sum_{j=0}^{l}a_j\left(\frac{1}{\cosh(\xi_n)+r}\right)^j+\sum_{j=1}^{l}b_j\left(\frac{1}{\cosh(\xi_n)+r}\right)^{(j-1)}\frac{\sinh(\xi_n)}{\cosh(\xi_n)+r}, \tag{6}$$

where $r$, $a_j$, $b_j$ $(j=0,\cdots,l)$ are constants to be determined and $l$ is given according to the homogeneous balance principle. From formula (2), we can obtain

$$u_{n+p_s}(\xi_n)=\sum_{j=0}^{l}a_j\left(\frac{1}{\cosh(\xi_n+\varphi_s)+r}\right)^j+\sum_{j=1}^{l}b_j\left(\frac{1}{\cosh(\xi_n+\varphi_s)+r}\right)^{(j-1)}\times\frac{\sinh(\xi_n+\varphi_s)}{\cosh(\xi_n+\varphi_s)+r}, \tag{7}$$

with $\varphi_s$ satisfying

$$\varphi_s=p_{s1}d_1+p_{s2}d_2+\cdots+p_{sQ}d_Q. \tag{8}$$

Meanwhile, it is important to note that $u_{n+p_s}(\xi_n)$ is a function of $\xi_n$ and not $\xi_{n+p_s}$.

**Step 3** Substituting the ansatzes (6) and (7) into Eq.(5), clearing the denominator and setting the coefficients of power terms in $\sinh(\xi_n)$ and $\cosh(\xi_n)$ to zero, a system of nonlinear algebraic equations with $a_j, b_j (j=0,\cdots,l)$ and $r$ are obtained. From these equations, we can obtain the corresponding undetermined coefficients.

Finally, substituting the obtained constants $a_j, b_j (j=0,\cdots,l)$ and $r$ into ansatz (6), a series of explicit and exact solutions can be obtained which include periodic wave solutions and solitary wave solutions.

## 3. Application to some special nonlinear equation

In this section, we apply the method developed in Section 2 to the discrete nonlinear (2+1)-dimensional Toda lattice equation and the discretized nonlinear mKdV lattice equation, and given some exact travelling wave solutions.

### 3.1. Discrete nonlinear (2+1)-dimensional Toda lattice equation

Let us consider the discrete nonlinear (2+1)-dimensional Toda lattice equation[22]

$$\frac{\partial^2 y_n}{\partial x\partial t}=\mathrm{e}^{(y_{n-1}-y_n)}-\mathrm{e}^{(y_n-y_{n+1})}, \tag{9}$$

where $y_n(x,t)$ is the displacement from equilibrium of the $n$th unit mass under an exponentially decaying interaction force between nearest neighbours. To write (9) as a polynomial DDE, set

$$\frac{\partial u_n}{\partial t}=\mathrm{e}^{(y_{n-1}-y_n)}-1. \tag{10}$$

Then (9) becomes

$$\frac{\partial^2 u_n}{\partial x \partial t} = \left(\frac{\partial u_n}{\partial t} + 1\right)(u_{n-1} - 2u_n + u_{n+1}). \qquad (11)$$

Let $u_n = u_n(\xi_n), \xi_n = kn + c_1 x + c_2 t + \delta$, then Eq.(11) becomes

$$c_1 c_2 u_n'' = (c_2 u_n' + 1)(u_{n-1} - 2u_n + u_{n+1}). \qquad (12)$$

We expand the solution of (12) in the form of (6) and (7). In this case, $p_s = 1, -1, d_1 = k$. According to the homogeneous balance principle, we can choose the solution of Eq.(12) in the form

$$u_n(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n) + r} + \frac{b_1 \sinh(\xi_n)}{\cosh(\xi_n) + r},$$

$$u_{n+1}(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n + k) + r} + \frac{b_1 \sinh(\xi_n + k)}{\cosh(\xi_n + k) + r},$$

$$u_{n-1}(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n - k) + r} + \frac{b_1 \sinh(\xi_n - k)}{\cosh(\xi_n - k) + r}. \qquad (13)$$

Substituting Eq.(13) into Eq.(12), clearing the denominator and setting the coefficients of power terms like $\sinh^i(\xi_n)\cosh^i(\xi_n)(i = 0, 1, 2, \cdots)$ to zero, a system of algebraic equations are obtained:

$$c_1 c_2 a_1 - 2a_1 \cosh(k) + 2a_1 = 0,$$

$$-2b_1 r + 2b_1 \cosh(k) r - c_1 c_2 b_1 r = 0,$$

$$\begin{aligned} &4c_2 b_1 a_1 r - c_1 c_2 a_1 r + 2c_1 c_2 a_1 r \cosh(k) + 2a_1 r \\ &- 4c_2 a_1 b_1 \cosh(k) r - 2a_1 \cosh(k) r = 0, \end{aligned}$$

$$\begin{aligned} &c_1 c_2 b_1 r^2 - 2c_1 c_2 b_1 \cosh(k) r^2 - 2c_2 a_1^2 \\ &+ 2b_1 \cosh(k) r^2 - 2b_1 r^2 - 2c_2 b_1^2 r^2 \\ &+ 2c_2 a_1^2 \cosh(k) + 2b_1 \cosh^2(k) \\ &- 2c_1 c_2 b_1 + 2c_2 b_1 \cosh^2(k) r^2 - 2b_1 = 0, \end{aligned}$$

$$\begin{aligned} &- 2c_1 c_2 a_1 \cosh(k) r^2 + 2a_1 \cosh^2(k) - 4c_2 b_1 a_1 r^2 \\ &- 2a_1 - 2a_1 r^2 - 2c_2 a_1 b_1 \cosh(k) + 4c_2 a_1 b_1 \cosh(k) r^2 \\ &- 3c_1 c_2 a_1 + 4c_2 b_1 a_1 + c_1 c_2 a_1 r^2 + c_1 c_2 a_1 \cosh^2(k) \\ &+ 2a_1 \cosh(k) r^2 - 2c_2 b_1 a_1 \cosh^2(k) = 0, \end{aligned}$$

$$\begin{aligned} &2c_2 b_1^2 r \cosh^2(k) - c_1 c_2 b_1 r^3 + 2b_1 r^3 \\ &- 2c_2 a_1^2 r \cosh(k) - 4b_1 r + 2c_1 c_2 b_1 \cosh(k) r^3 \\ &- c_1 c_2 b_1 r \cosh^2(k) + c_1 c_2 b_1 r \\ &- 4c_2 b_1 r^2 + 4b_1 r \cosh^2(k) + 2c_2 a_1^2 r \\ &- 4c_1 c_2 b_1 r \cosh(k) + 2c_2 b_1^2 r^3 - 2b_1 \cosh(k) r^3 \\ &+ 2c_2 b_1^2 r \cosh(k) - 2c_2 b_1^2 \cosh(k) r = 0, \end{aligned}$$

$$\begin{aligned} &4a_1 r \cosh^2(k) - 4c_1 c_2 a_1 r \cosh(k) + 4c_2 a_1 b_1 \cosh(k) r \\ &+ 2a_1 \cosh(k) r^3 - 2a_1 r^3 - c_1 c_2 a_1 r \cosh^2(k) \\ &+ c_1 c_2 a_1 r + 2c_2 b_1 a_1 r \cosh^2(k) - 4a_1 r \\ &- c_1 c_2 a_1 r^3 - 6c_2 b_1 a_1 r = 0, \end{aligned}$$

$$\begin{aligned} &2c_2 a_1^2 - 3c_1 c_2 b_1 r^2 - 2c_2 b_1^2 \cosh(k) r^2 \\ &+ 2b_1 r^4 - 2b_1 r^2 - 2c_1 c_2 b_1 \cosh^2(k) \\ &- 2c_2 a_1^2 \cosh^2(k) + 2c_2 b_1^2 r^2 \\ &- 2b_1 \cosh(k) r^4 + c_1 c_2 b_1 r^2 \cosh^2(k) \\ &+ 2b_1 r^2 \cosh^2(k) + c_1 c_2 b_1 r^4 - 2c_2 b_1^2 \\ &+ 2c_1 c_2 b_1 + 2c_2 b_1^2 \cosh^2(k) = 0, \end{aligned}$$

$$\begin{aligned} &- 2c_2 a_1 b_1 \cosh(k) r^2 - 2c_1 c_2 a_1 \cosh^2(k) \\ &+ 2c_2 b_1 a_1 r^2 + 2c_1 c_2 a_1 - 2a_1 r^2 \\ &+ 4c_2 b_1 a_1 \cosh^2(k) - 4c_2 b_1 a_1 \\ &+ 2a_1 r^2 \cosh^2(k) - 2c_1 c_2 a_1 r^2 = 0. \end{aligned} \qquad (14)$$

From above, we obtain five types of solutions:

$$r = 0, \quad a_0 = a_0, \quad a_1 = 0,$$

$$b_1 = \frac{\sinh^2(k)}{c_2}, \quad c_1 = \frac{\sinh^2(k)}{c_2}, \quad c_2 = c_2;$$

$$r = 0, \quad a_0 = a_0, \quad a_1 = \pm\frac{I(\cosh(k) - 1)}{c_2},$$

$$b_1 = \frac{\cosh(k) - 1}{c_2}, \quad c_1 = \frac{2(\cosh(k) - 1)}{c_2}, \quad c_2 = c_2;$$

$$a_0 = a_0, \quad a_1 = 0, \quad r = 1, \quad b_1 = \frac{\cosh(k) - 1}{c_2},$$

$$c_1 = \frac{2(\cosh(k) - 1)}{c_2}, \quad c_2 = c_2;$$

$$a_0 = a_0, \quad a_1 = 0, \quad r = -1, \quad b_1 = \frac{\cosh(k) - 1}{c_2},$$

$$c_1 = \frac{2(\cosh(k) - 1)}{c_2}, \quad c_2 = c_2;$$

where $I^2 = -1$, $a_0, k, c_2$ are arbitrary constants; and

$$b_1 = b_1, \quad r = r, \quad a_0 = a_0, \quad a_1 = \pm\sqrt{(r^2 - 1)} b_1,$$

$$c_1 = 2b_1, c_2 = \frac{\cosh(k) - 1}{b_1},$$

where $a_0$, $k$, $b_1$ are arbitrary constants, $r^2 \geq 1$. Then the closed form solutions of (11) are

$$u_{n1} = a_0 + \frac{\sinh^2(k) \tanh\left(kn + \frac{x \sinh^2(k)}{c_2} + c_2 t + \delta\right)}{c_2}, \qquad (15)$$

$$u_{n2,2'} = a_0 \pm \frac{I(\cosh(k)-1)}{c_2}\operatorname{sech}\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right) + \frac{(\cosh(k)-1)\tanh\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right)}{c_2}, \tag{16}$$

$$u_{n3} = a_0 + \frac{(\cosh(k)-1)\sinh\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right)}{c_2\left(1 + \cosh\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right)\right)}, \tag{17}$$

$$u_{n4} = a_0 + \frac{(\cosh(k)-1)\sinh\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right)}{c_2\left(\cosh\left(kn + \frac{2x(\cosh(k)-1)}{c_2} + c_2 t + \delta\right) - 1\right)}, \tag{18}$$

$$u_{n5,5'} = a_0 \pm \frac{\sqrt{(r^2-1)}b_1}{r + \cosh\left(kn + 2b_1 x + \frac{t(\cosh(k)-1)}{b_1} + \delta\right)} + \frac{b_1\sinh\left(kn + 2b_1 x + \frac{t(\cosh(k)-1)}{b_1} + \delta\right)}{r + \cosh\left(kn + 2b_1 x + \frac{t(\cosh(k)-1)}{b_1} + \delta\right)}, \tag{19}$$

where $u_{n1}$ is a kink-type excitation and the others are new travelling wave solutions.

### 3.2. Discretized nonlinear mKdV lattice equation

The discretized mKdV lattice equation[23] reads

$$\frac{\partial u_n(t)}{\partial t} = (\alpha - u_n^2)(u_{n+1} - u_{n-1}), \tag{20}$$

where $\alpha$ is a constant. Let $u_n = u_n(\xi_n), \xi_n = kn + ct + \delta$, then Eq.(20) becomes

$$cu_n' = (\alpha - u_n^2)(u_{n+1} - u_{n-1}). \tag{21}$$

We expand the solution of (21) in the form of (6) and (7). In this case, $p_s = 1, -1, d_1 = k, c_1 = c$. According to the homogeneous balance principle, we can choose the solution of Eq.(21) in the form

$$u_n(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n) + r} + \frac{b_1\sinh(\xi_n)}{\cosh(\xi_n) + r},$$

$$u_{n+1}(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n + k) + r} + \frac{b_1\sinh(\xi_n + k)}{\cosh(\xi_n + k) + r},$$

$$u_{n-1}(\xi_n) = a_0 + \frac{a_1}{\cosh(\xi_n - k) + r} + \frac{b_1\sinh(\xi_n - k)}{\cosh(\xi_n - k) + r}. \tag{22}$$

Substituting Eq.(22) into Eq.(21), clearing the denominator and setting the coefficients of power terms like $\sinh^i(\xi_n)\cosh^i(\xi_n)(i = 0, 1, 2, \cdots)$ to zero, yields the following algebraic equations:

$$\begin{aligned}
&-2\alpha a_2\sinh(k)r + ca_2 r - 4a_1 a_0 a_2\sinh(k) \\
&+ 2a_2^3\sinh(k)r + 2a_0^2 a_2\sinh(k)r = 0,
\end{aligned}$$

$$\begin{aligned}
&-2a_2 a_1^2\sinh(k) - 2a_0 a_1^2\sinh(k) + 4a_2^2 a_0\sinh(k)r \\
&- ca_1 + 2\alpha a_1\sinh(k) = 0,
\end{aligned}$$

$$\begin{aligned}
&2a_0^2 a_2\cosh(k)\sinh(k) + 4a_0^2 a_2\sinh(k)r^2 + ca_2 \\
&+ 2a_2^2\cosh(k)\sinh(k) + 2ca_2 r^2\cosh(k) \\
&- 4a_1^2 a_2\sinh(k) - 2\alpha a_2\cosh(k)\sinh(k) \\
&- 4\alpha a_2\sinh(k)r^2 = 0,
\end{aligned}$$

$$\begin{aligned}
&4a_1 a_2^2\sinh(k)r + 4a_2^2 a_0\sinh(k)r^2 \\
&+ 4\alpha a_1 r\sinh(k) - 2ca_1 r\cosh(k) \\
&+ 4a_2^2 a_0\cosh(k)\sinh(k) - 4a_1^2 a_0\sinh(k) \\
&- 4a_0^2 a_1 r\sinh(k) = 0,
\end{aligned}$$

$$
\begin{aligned}
&-ca_2r+ca_2r^3+4a_1a_0a_2\sinh(k)+ca_2r\cosh^2(k)\\
&-2a_2^3\sinh(k)r+4a_0^2a_2\cosh(k)r\sinh(k)\\
&-2\alpha a_2\sinh(k)r^3+2a_1^2a_2r\sinh(k)\\
&+2a_0^2a_2\sinh(k)r^3+4a_1a_0a_2\cosh(k)\sinh(k)\\
&+4a_2a_0a_1r^2\sinh(k)+2ca_2r\cosh(k)\\
&-4\alpha a_2\cosh(k)r\sinh(k)=0,
\end{aligned}
$$

$$
\begin{aligned}
&-2a_1^3\sinh(k)+4a_2^2a_0\sinh(k)r\cosh(k)\\
&-ca_1r^2-4a_1^2a_0r\sinh(k)+2\alpha a_1r^2\sinh(k)\\
&-2a_0^2a_1r^2\sinh(k)+ca_1+2a_2^2a_1\sinh(k)\\
&+4a_1a_2^2\sinh(k)\cosh(k)-ca_1\cosh^2(k)=0,
\end{aligned}
$$

$$
\begin{aligned}
&ca_2r^2-2\alpha a_2\sinh(k)r^2\cosh(k)-ca_2\\
&+4a_1a_0a_2\sinh(k)r\cosh(k)+ca_2\cosh^2(k)\\
&+2a_1^2a_2\sinh(k)\cosh(k)\\
&-2a_2^3\cosh(k)\sinh(k)+4a_1a_0a_2\sinh(k)r\\
&+2a_0^2a_2\sinh(k)r^2\cosh(k)+4a_1^2a_2\sinh(k)=0. \quad (23)
\end{aligned}
$$

We solve the above system, and obtain the solutions

$$c=2\alpha\sinh(k),\ r=0,\ a_0=0,$$
$$a_1=\pm\sqrt{\alpha(1-\cosh^2(k))},\ b_1=0;$$

$$c=\frac{2\sinh(k)\alpha(2r^2-\cosh(k)-1)}{(\cosh(k)+1)(r^2-1)},\ r=r,$$
$$a_0=\pm\frac{r\sqrt{\alpha(\cosh(k)-1)}}{(\cosh(k)+1)(r^2-1))},$$
$$a_1=\mp\frac{\alpha(2r^2\cosh(k)-2r^2+1-\cosh^2(k))}{\sqrt{\alpha(\cosh(k)-1)(\cosh(k)+1)(r^2-1)}},$$
$$b_1=0;$$

$$a_1=0,\ c=2\alpha\tanh(2k),\ r=0,\ a_0=0,$$
$$b_1=\pm\sqrt{\alpha}\tanh(2k);$$
$$a_1=\pm\frac{\sqrt{(\alpha(r^2-1)}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)},\ c=\frac{4\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)},$$
$$a_0=0,\ r=r,\ b_1=\pm\frac{\sqrt{\alpha}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)};$$
$$a_1=\pm\frac{\sqrt{-\alpha}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)},\ c=\frac{4\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)},$$
$$r=0,\ a_0=0,\ b_1=\pm\frac{\sqrt{\alpha}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}.$$

where $k$ is an arbitrary constant. Then, the closed form solutions of (20) are:

$$u_{n1,1'}=\pm\sqrt{\alpha(1-\cosh^2(k))}\mathrm{sech}(kn+2\alpha\sinh(k)t+\delta), \quad (24)$$

which is a new bell-type solitary solution;

$$
\begin{aligned}
u_{n2,2'}=&\pm\frac{r\sqrt{\alpha(\cosh(k)-1)}}{(\cosh(k)+1)(r^2-1)}\\
&\mp\frac{\alpha(2r^2\cosh(k)-2r^2+1-\cosh(k)^2)}{\sqrt{\alpha(\cosh(k)+1)(\cosh(k)+1)(r^2-1)}(r+\cosh(kn+\frac{2t\sinh(k)\alpha(2r^2-\cosh(k)-1)}{(\cosh(k)+1)(r^2-1)}+\delta))}, \quad (25)
\end{aligned}
$$

which is a new travelling wave solution;

$$u_{n3,3'}=\pm\sqrt{\alpha}\tanh(2k)\tanh(kn+2t\alpha\tanh(2k)+\delta), \quad (26)$$

which is a new kink-type solitary solution;

$$u_{n4,4'}=\pm\frac{\sqrt{\alpha(r^2-1)}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)(r+\cosh(kn+\frac{4t\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}+\delta)}\pm\frac{\sqrt{(\alpha)}(\mathrm{e}^k-1)\sinh(kn+\frac{4t\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}+\delta)}{(\mathrm{e}^k+1)(r+\cosh(kn+\frac{4t\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}+\delta))}, \quad (27)$$

which is a new travelling wave solution; and

$$u_{n5,5'}=\pm\frac{\sqrt{-\alpha}(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}\mathrm{sech}(kn+\frac{4t\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}+\delta)\pm\frac{\sqrt{(\alpha)}(\mathrm{e}^k-1)\tanh(kn+\frac{4t\alpha(\mathrm{e}^k-1)}{(\mathrm{e}^k+1)}+\delta)}{(\mathrm{e}^k+1)}, \quad (28)$$

which is a new travelling wave solutions.

## 4. Summary

In this paper,we have successfully used the hyperbolic function approach to construct solutions for discrete nonlinear (2+1)-dimension Toda lattice equation and discrete mKdV lattice equation, obtained new bell-type solitary solution, kink-type solitary solution and abundant new travelling wave solutions. Not only has it obtained many kinds of solutions, but also the procedure is readily computable. Therefore, in view of the importance of the DDEs, the proposed method is worthy of further study and it may help us to find new and interesting solutions for a given nonlinear discrete system.

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