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Unified symmetry of holonomic mechanical systems*

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The definition and the criterion of a unified symmetry for a holonomic mechanical system are presented based on the total time derivative along the trajectory of the system. A new conserved quantity, as well as the Noether conserved quantity and the Hojman conserved quantity, deduced from the unified symmetry, is obtained. An example is given to illustrate the application of the results.

Keywords: holonomic mechanical system, unified symmetry, conserved quantity
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1. Introduction

The conservation laws for a dynamical system play an important role in the fields of mathematics, mechanics and physics. The symmetry method is a modern method for finding conserved quantities of constrained mechanical systems. There are mainly three kinds of symmetries: namely, the Noether symmetry,[1−7] the Lie symmetry[8−13] and the form invariance.[14−17] And there are mainly three kinds of conserved quantities: i.e. the Noether conserved quantity,[1−7] the Hojman conserved quantity[18,19] and a new conserved quantity,[20−23] which can, respectively, be deduced from the Noether symmetry, the special Lie symmetry and the form invariance.

Mei *et al* gave a unified symmetry[24] of a Lagrangian system which could lead to the three kinds of conserved quantities mentioned above. In this paper, on the basis of the total time derivative along the trajectory of the system, we generalize Mei's results to a holonomic mechanical system and obtain the three kinds of conserved quantities mentioned above.

2. Definition and criterion of unified symmetry

Consider a non-singular holonomic mechanical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s \qquad (s=1,2,\cdots,n), \tag{1}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian of the system, and $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ are the generalized non-potential forces. From Eq.(1), we can find all of the generalized accelerations as

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad (s=1,2,\cdots,n). \tag{2}$$

Definition[24] If a symmetry is a Noether one as well as a Lie one and is also form invariant at one time, the symmetry is called a unified symmetry.

For the holonomic mechanical system (1), the Noether identity can be written in the form

$$L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+\mathrm{X}^{(1)}(L)+\sum_{s=1}^{n}Q_s(\xi_s-\dot{q}_s\xi_0)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}=0 \tag{3}$$

on the basis of the total time derivative along the trajectory of the system, where

$$\begin{aligned}\mathrm{X}^{(1)}=&\xi_0\frac{\partial}{\partial t}+\sum_{s=1}^{n}\xi_s\frac{\partial}{\partial q_s}\\&+\sum_{s=1}^{n}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s},\end{aligned} \tag{4}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\sum_{s=1}^{n}\dot{q}_s\frac{\partial}{\partial q_s}+\sum_{s=1}^{n}\alpha_s\frac{\partial}{\partial \dot{q}_s}. \tag{5}$$

ξ_0 and ξ_s are the infinitesimal generators of transformations and $G_{\mathrm{N}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is a gauge function. The

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determining equations of the Lie symmetry have the form

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \mathrm{X}^{(1)}(\alpha_s) \quad (s=1,2,\cdots,n). \tag{6}$$

The criterion equations of the form invariance are

$$\bar{\mathrm{E}}_s[\mathrm{X}^{(1)}(L)] = \mathrm{X}^{(1)}(Q_s) \qquad (s=1,2,\cdots,n), \tag{7}$$

where

$$\bar{\mathrm{E}}_s = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s} - \frac{\partial}{\partial q_s} \qquad (s=1,2,\cdots,n). \tag{8}$$

Using the definition and relations (3), (6) and (7), we have:

Criterion If there exists such a gauge function $G_{\mathrm{N}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ that the infinitesimal generators ξ_0 and ξ_s satisfy the condition

$$\begin{aligned}&\left[L\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \mathrm{X}^{(1)}(L) + \sum_{s=1}^{n}Q_s(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}}\right]^2 \\ &+\sum_{s=1}^{n}\{\bar{\mathrm{E}}_s[\mathrm{X}^{(1)}(L)] - \mathrm{X}^{(1)}(Q_s)\}^2 \\ &+\sum_{s=1}^{n}\left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 \right.\\ &\left. - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \mathrm{X}^{(1)}(\alpha_s)\right]^2 = 0,\end{aligned} \tag{9}$$

then the symmetry is a unified symmetry of the holonomic mechanical system.

3. Conserved quantities deduced from unified symmetry

The unified symmetry of the holonomic mechanical system can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity under certain conditions.

Proposition 1 For the holonomic mechanical system, the unified symmetry will lead to the Noether conserved quantity

$$I_{\mathrm{N}} = L\xi_0 + \sum_{s=1}^{n}\frac{\partial L}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{N}} = \mathrm{const.} \tag{10}$$

Proof Because the unified symmetry is certainly a Noether symmetry, there exists such a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ that the Noether identity (3) holds. According to Ref.[22], the system has the conserved quantity (10).

Proposition 2 For the holonomic mechanical system, under the special infinitesimal transformations in which time is not varied, if there exists such a function $\mu = \mu(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ that

$$\sum_{s=1}^{n}\frac{\partial\alpha_s}{\partial\dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{11}$$

then the unified symmetry will lead to the Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu}\sum_{s=1}^{n}\left[\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{\partial}{\partial\dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\right] = \mathrm{const.} \tag{12}$$

Proof Because the unified symmetry is certainly a Lie symmetry, so we have the determining equations of the special Lie symmetry ($\xi_0 = 0$)

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \sum_{k=1}^{n}\frac{\partial\alpha_s}{\partial q_k}\xi_k + \sum_{k=1}^{n}\frac{\partial\alpha_s}{\partial\dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k \quad (s=1,2,\cdots,n). \tag{13}$$

Proposition 2 is easily verified by using relations (11) and (13).

Proposition 3 For the holonomic mechanical system, if there exists such a function $G_{\mathrm{F}} = G_{\mathrm{F}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ that

$$\begin{aligned}&\mathrm{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \mathrm{X}^{(1)}[\mathrm{X}^{(1)}(L)] \\ &+\sum_{s=1}^{n}\mathrm{X}^{(1)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} = 0,\end{aligned} \tag{14}$$

then the unified symmetry will lead to the new conserved quantity

$$\begin{aligned}I_{\mathrm{F}} =& \mathrm{X}^{(1)}(L)\xi_0 + \sum_{s=1}^{n}\frac{\partial\mathrm{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{F}} \\ =& \mathrm{const.}\end{aligned} \tag{15}$$

Proof Because the unified symmetry is certainly a form invariance, so the criterion equations (7) hold. Using relations (7) and (14), we can prove that proposition 3 is true.[20]

4. An illustrative example

The Whittaker equation[25]

$$\ddot{q}_1 = q_1, \quad \ddot{q}_2 = \dot{q}_1 \tag{16}$$

can be written in the form of a holonomic mechanical system with

$$\begin{aligned}L =& \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_1^2, \\ Q_1 =& 0, \quad Q_2 = \dot{q}_1.\end{aligned} \tag{17}$$

Choose the infinitesimal generators as

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = t. \tag{18}$$

After calculation, we have

$$\begin{aligned} &\mathrm{X}^{(1)}(L) = \dot{q}_2, \quad \mathrm{X}^{(1)}[\mathrm{X}^{(1)}(L)] = 1, \\ &\mathrm{X}^{(1)}(Q_1) = 0, \quad \mathrm{X}^{(1)}(Q_2) = 0, \\ &\mathrm{X}^{(1)}(\alpha_1) = 0, \quad \mathrm{X}^{(1)}(\alpha_2) = 0. \end{aligned} \tag{19}$$

It is easy to verify that the generators (18) are unified-symmetrical.

Substituting relations (17), (18) and (19) into the Noether identity (3), we obtain the gauge function

$$G_{\mathrm{N}} = \dot{q}_1 - tq_1 - q_2, \tag{20}$$

and proposition 1 gives

$$I_{\mathrm{N}} = \dot{q}_1 + t\dot{q}_2 - tq_1 - q_2 = \mathrm{const}, \tag{21}$$

which is a Noether conserved quantity deduced from the unified symmetry (18).

Formula (11) becomes

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{22}$$

which has the solution

$$\mu = q_1 - \dot{q}_2. \tag{23}$$

From relations (18) and (23), using proposition 2, we obtain

$$I_{\mathrm{H}} = -(q_1 - \dot{q}_2)^{-1} = \mathrm{const}, \tag{24}$$

which is a Hojman conserved quantity deduced from the unified symmetry (18).

Substituting relations (18) and (19) into Eq.(14), we obtain

$$G_{\mathrm{F}} = -t + (q_1 - \dot{q}_1)\exp\, t, \tag{25}$$

and proposition 3 gives

$$I_{\mathrm{F}} = (q_1 - \dot{q}_1)\exp\, t = \mathrm{const}, \tag{26}$$

which is a new conserved quantity deduced from the unified symmetry (18).

It is obvious that both the Hojman conserved quantity (24) and the new conserved quantity (26) are non-Noether conserved quantities.

5. Conclusion

The unified symmetry shown in the paper is a common symmetry of three kinds of symmetries. Three main conserved quantities, i.e. the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity, are found by using the unified symmetry for the holonomic mechanical system. We expect to deduce more other kinds of non-Noether conserved quantities from the unified symmetry. Maybe the results can be generalized to a non-holonomic mechanical system.

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