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Chaos in Bianchi I cosmology*

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We have investigated the dynamical evolution of the Yang–Mills (YM) field in Bianchi I cosmology background. We find that the long-time evolution behaviour of the YM field is highly sensitive to initial conditions, i.e. small fluctuations of initial conditions for fixed Hamiltonian of the system may rapidly change the evolution of the field. By using the Poincaré section method, we further illustrate that the dynamical evolution of the YM field in Bianchi I cosmology background has certain typically chaotic properties.

Keywords: chaos, Poincaré section method, dynamical evolution
PACC: 0314

1. Introduction

The evolution of the early universe is an enigma. Since Lorenz found the deterministic non-periodic flow,[1] chaotic behaviour was considered as an interesting phenomenon by many physicists. In the past decades, chaos is a most important idea used to explain various nonlinear phenomena in nature. After the research on the three-body problem by Poincaré, many studies about chaos in the celestial mechanics and astrophysics have been done and also have found the important role of chaos in the universe.[2−10] There are some disputes on chaos of the Mixmaster dynamics in Refs.[11,12]. Latifi *et al*[13] and Contopoulos *et al*[14] have shown the nonintegrability of the Mixmaster model in the Painlevé sense, although the question of the generic behaviour (chaotic or not) remained unsettled mainly due to the absence of an invariant (or topological) characterization of chaos in the models.[12,15] Recently, Cornish and Levin[16,17] proposed to quantify the chaos in the Mixmaster universe by calculating the dimensions of fractal basin boundaries in initial condition sets for the full dynamics. Some cosmologists are interested in the effects of anisotropy on the dynamics of the early universe, because they might hope to gain a better understanding of the behaviour of the model that generalizes an isotropic mini-super-space to a full super-space by adding more degrees of freedom. Bianchi cosmologies with fluid sources are such models. The matter in these models is either a perfect fluid[18] or that consists of massive or massless vector fields.[19,20] Oliveira *et al*[21−24] have studied the chaotic behaviour in the Bianchi IX model. Due to the strong instability of the trajectories with respect to small changes of initial conditions, the authors of Refs.[25,26] drew a conclusion on non-integrability and stochasticity of the Yang–Mills classical mechanics (YMCM) system. Savvidy[27] investigated the statistical properties of the YMCM from the viewpoint of ergodic theory.[28−30] There are extensive works on various Friedmann–Robertson–Walker cosmology with Yang–Mills (YM) field source that has a stress–energy tensor of the form of a trace-free perfect fluid.[31−35] We expect to yield a better understanding of the vacuum state in YM theory by the non-perturbative treatment of homogenous source-free YM field as a dynamical system, just like a non-perturbative mini-super-space Einstein–Yang–Mills theory results in a better understanding of the vacuum state of YM field in the Planck regime.

In this paper, we investigate the dynamical properties of the axially symmetric YM field in Bianchi 1 cosmology. This paper is organized as follows. In

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Section 2, we establish the equation of motion of the axially symmetric YM field in Bianchi I cosmology and the Hamiltonian constraint of the dynamical system. In Section 3, we investigate the stability of the system with the nonlinear theory and we also study the evolution of the field in the phase space. In Section 4, we further investigate the evolution properties of the YM field with Pioncaré section, and find the chaotic evolution of the system. A brief conclusion is given in the last section.

2. Stability analysis of the axially symmetric YM field in Bianchi I cosmology

The dynamics of axially symmetric YM field in Bianchi I cosmology[36] is governed by

$$\ddot{\alpha} + \alpha(\phi^2 + \alpha^2) = 0, \tag{1}$$

$$\ddot{\phi} + 2\alpha^2\phi = 0, \tag{2}$$

which correspond to a set of two strongly coupled oscillators with varying frequencies and amplitudes. These equations are derived from the Hamiltonian constraint

$$H = \frac{1}{2}\left(P_\phi^2 + \frac{1}{2}P_\alpha^2 + 2\alpha^2\phi^2 + \alpha^4\right), \tag{3}$$

where P_ϕ and P_α are the momenta canonically conjugated to ϕ and α respectively. The system describes the motion of a point particle moving in a potential well $2\alpha^2\phi^2 + \alpha^4$ with two open channels in the directions of positive and negative ϕ. The complete dynamics of the system is described by the following Hamiltonian equation:

$$\dot{\phi} = P_\phi, \tag{4}$$

$$\dot{\alpha} = \frac{1}{2}P_\alpha, \tag{5}$$

$$\dot{P}_\phi = -2\phi\alpha^2, \tag{6}$$

$$\dot{P}_\alpha = -2(\alpha\phi^2 + \alpha^3). \tag{7}$$

The above dynamical system in the phase space has a critical point whose coordinates are

$$P_\phi = 0,\ P_\alpha = 0,\ \alpha = 0,\ \phi = c(\text{constant}). \tag{8}$$

The stability analysis is obtained by linearizing the dynamical system of Eq.(4–7) about the critical point (8). We have

$$\begin{pmatrix} \dot{\phi} \\ \dot{\alpha} \\ \dot{P}_\phi \\ \dot{P}_\alpha \end{pmatrix} = S \begin{pmatrix} \phi \\ \alpha \\ P_\phi \\ P_\alpha \end{pmatrix}, \tag{9}$$

where S is the linearizing matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c^2 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{10}$$

The matrix has eigenvalues

$$\lambda_{1,2} = 0, \lambda_{3,4} = \pm\sqrt{2}c, \tag{11}$$

where the second pair of real eigenvalues generate a pure saddle structure which reveals a manifold of the linearized unstable orbit.[37]

In order to investigate the dynamical evolution of the YM field in the Bianchi I cosmology background and its chaotic dynamical behaviours, we use variables $(P_\alpha, P_\phi, \alpha, \phi)$ and the package Poincaré[38] to perform the following numerical simulations. In Fig.1 with $H = 0.07$, we illustrate numerically that the non-integrability of the system results in the distortion and twisting of the orbits in the phase space. These unstable orbits cross each other transversally, produce chaotic sets[39,40] and consequently the "destroyed" regions of the Poincaré sections of the system, as shown in the following section. In Fig.1 we show the evolution of the orbit in three-dimension (α, P_α, ϕ) and different two-dimensional planes. We can see the complexity of unstable evolution of the orbit in three-dimensional phase space. In order to see the evolution of the orbit, we project it into different two-dimensional planes. We can see, from (P_α, ϕ) and (α, ϕ) planes, that the orbit oscillates with unstable period, which means that the evolution of the orbit is stochastic; at the same time we find similar properties in the other planes, such as (α, P_α), (P_α, P_ϕ) and (ϕ, P_ϕ).

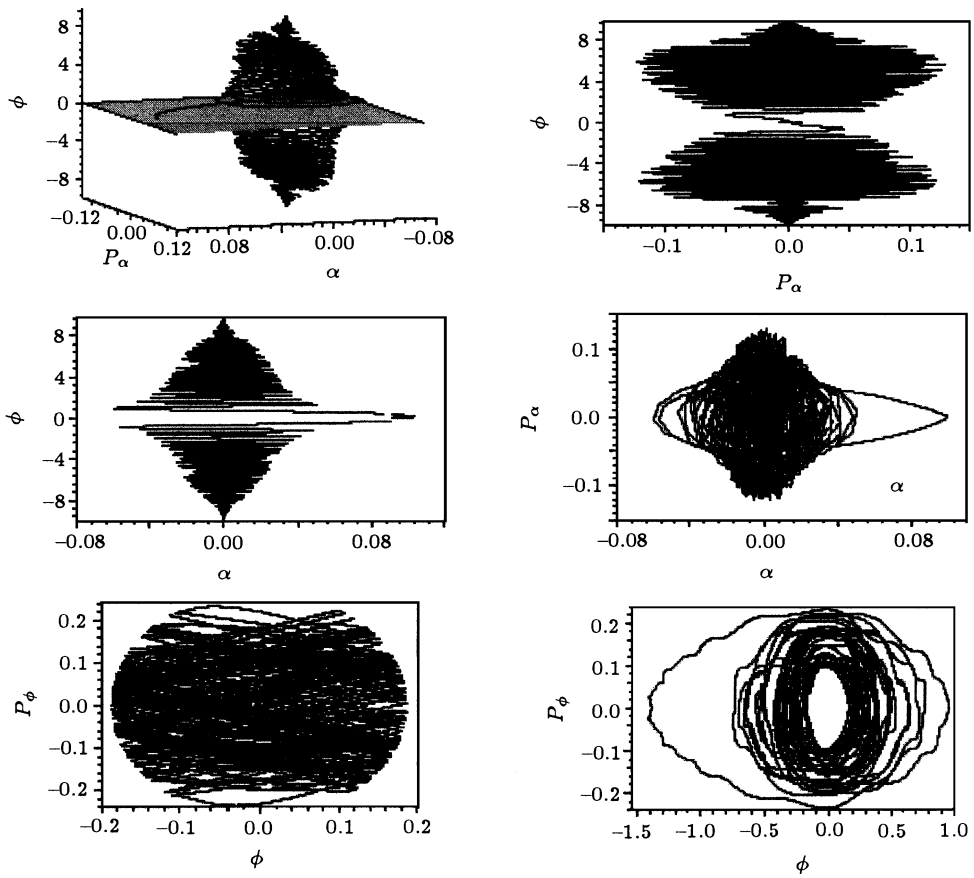


Fig.1. The orbital evolution of the YM field in three-dimension (α, P_α, ϕ) and different two-dimensional planes. We can see the complexity of unstable evolution of the orbit in three-dimensional phase space. In order to see the evolution of the orbit, we project it into different two-dimensional planes. We can see, from the (P_α, ϕ) and (α, ϕ) plane, that the orbit oscillates with an unstable period, which means that the evolution of the orbit is stochastic, at the same time we find similar properties in the other plane such as (α, P_α), (P_α, P_ϕ) and (ϕ, P_ϕ).

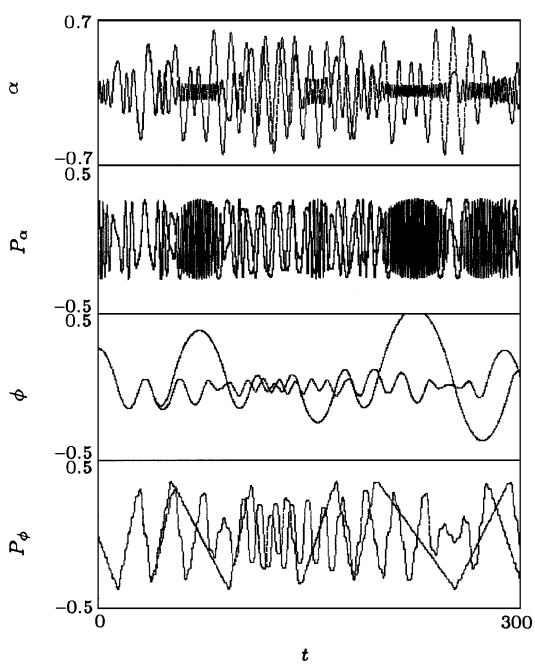


Fig.2. The long-time behaviour of the evolution of the YM field for $H = 0.07$. It is highly sensitive to initial conditions: small fluctuations of initial conditions for fixed H may rapidly change the evolution of the YM field.

As a consequence of non-integrability, the long-time behaviour of the evolution will be highly sensitive to initial conditions: small fluctuations of initial conditions for fixed H may rapidly change the evolution of the field. This behaviour is associated with the existence of chaotic sets generated by homo-clinic phenomena, due to the transversal crossings of homo-clinic cylinders in the phase space. In Fig.2, we draw the evolution of the field for two different initial conditions with fixed $H = 0.07$, we can see that the evolution is unstable and sensitively depends on initial conditions. It is clear that the amplitude of the oscillatory motion is inversely proportional to the frequency of the oscillations. There are typical chaotic evolution properties in the phase space and we will further illustrate in the next section with the Poincaré section method.

3. Further numerical results

To determine the chaotic behaviour of a dynamical system, we can further perform the Poincaré sec-

tion in the phase space. If the motion is not chaotic, the plotted points form a closed curve in the two-dimensional plane, because a regular orbit will move on a torus in the phase space and the curve is a cross section of the torus. If the orbit is chaotic, some of those tori will be broken and the Poincaré section does not consist of a set of closed curves, but the points will be distributed randomly in the allowed region to form a chaotic sea, just like Arnold diffusion.[41] From the distribution of the points in the Poincaré section, we can judge whether the motion is chaotic or not. In Fig.3 we perform Poincaré sections for different values of H in the section $\phi = 0, \dot{\phi} > 0$(the left column) and in the section $\alpha = 0, \dot{\alpha} > 0$ (the right column). We exhibit the Poincaré sections of the system for $H = 0.04$ in the first row of Fig.3, the sections make explicit the coexistence of Kolmogorov–Arnold–Moser (KAM) tori and destroyed regions in the phase space, as a consequence of the non-integrability of the system. If we increase the value of H ($H = 0.05$) for the second row of Fig.3, we find that the coexistence of KAM tori and destroyed regions is similar to the first row, but we also find that the destroyed region, i.e. the chaotic sea becomes larger, that is to say, the system becomes even non-integrability and its evolution becomes stochastic. If we further change the parameters and the initial conditions to draw Poincaré sections (the third row) for $H = 0.07$, a further larger chaotic sea will be obtained in an allowed region and the islands will be almost submerged by the chaotic sea. This means that the quasi-periodic orbits change into chaos.

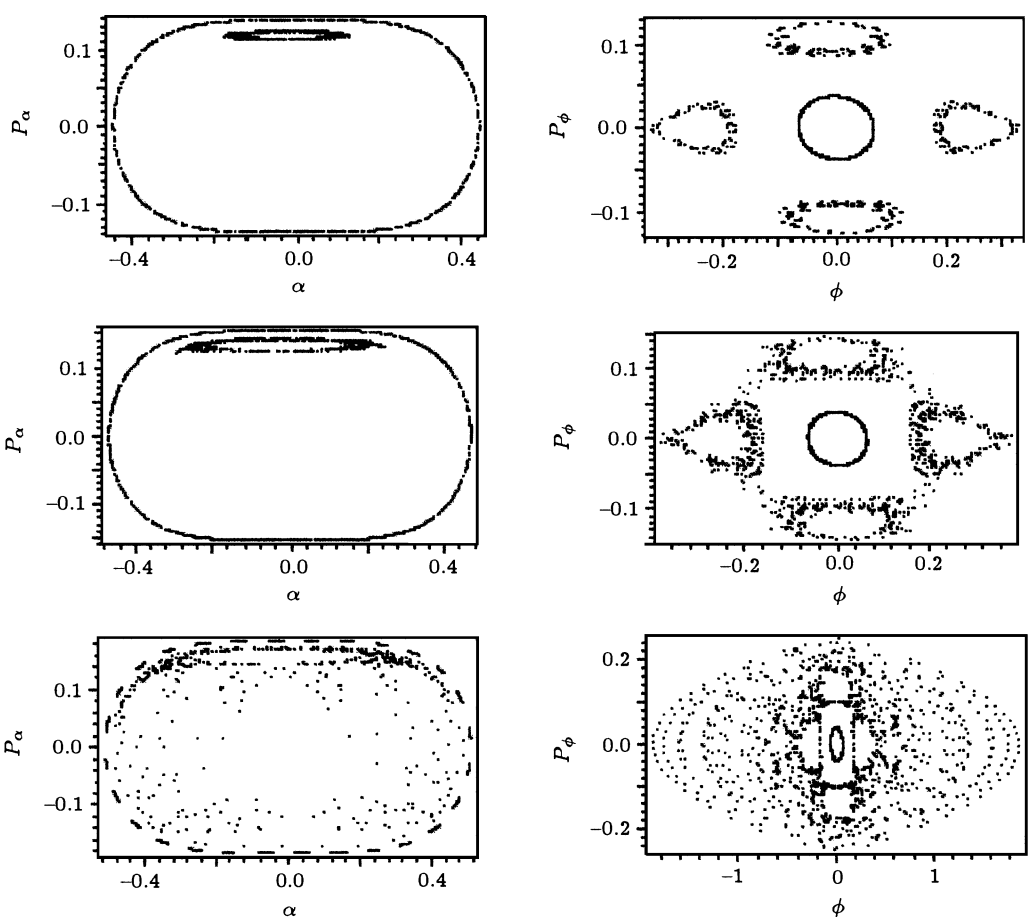


Fig.3. Poincaré sections for different values of H: in the section $\phi = 0, \dot{\phi} > 0$(the left column) and in the section $\alpha = 0, \dot{\alpha} > 0$ (the right column). Poincaré sections of the system for $H = 0.04$ are shown in the first row and $H = 0.05$ in the second row. In Poincaré sections (the third row) for $H = 0.07$, a further larger chaotic sea will be obtained in an allowed region, and the islands will be almost submerged by the chaotic sea, indicating that the quasi-periodic orbit changes into chaos.

4. Conclusions

We have investigated the dynamical evolution of the YM field in Bianchi I cosmology background, and find that the long-time behaviour of the evolution is highly sensitive to initial conditions, i.e. small fluctuations of initial conditions for a fixed H may rapidly change the evolution of the field. By using the Poincaré section method, we further illustrate that the dynamical evolution of the YM field in Bianchi I cosmology background has certain typically chaotical properties.

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