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Thermal entanglement in 1D optical lattice chains with nonlinear coupling*

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The thermal entanglement of spin-1 atoms with nonlinear coupling in an optical lattice chain is investigated for two-particle and multi-particle systems. It is found that the relation between linear coupling and nonlinear coupling is the key to determine thermal entanglement, which shows in what kinds of atoms thermal entanglement exists. This result is true both for two-particle and multi-particle systems. For multi-particle systems, the thermal entanglement does not decrease greatly, and the critical temperature decreases only slightly.

Keywords: optical lattice chain, nonlinear coupling, thermal entanglement
PACC: 4250, 3280

1. Introduction

Entanglement as a valuable resource for quantum information processing (QIP)[1,2] has attracted attention in recent years in both experimental and theoretical researches.[3−5] Because the entanglement is very fragile, the problem of how to create stable entanglement remains a main focus of recent studies in the field of quantum information processing. Thermal entanglement, which differs from other entanglement by its advantages of stability and requiring neither measurement nor controlled switching of interactions in the preparation process, is an attractive topic that has been extensively studied in the isotropic Heisenberg chain,[6−8] anisotropic Heisenberg chain,[9] and Ising model[10] since the seminal works of by Arnesen *et al*[6] and Nielsen.[11] Recently, Mancini and Bose have shown that the Ising model can be realized in cavity-QED (cavity quantum electrodynamics).[12]

The system of atoms in optical lattices is among the promising candidates for quantum information processing. It can take advantage of the technology used in atom optics and laser cooling based on the optical manipulation of atoms.[13] It also has the capability of being scaled and miniaturized in QIP.

In this paper, we study the thermal entanglement in optical lattice with nonlinear coupling for two-particle and multi-particle systems. We calculate the thermal entanglement as a function of the nonlinear coupling constant K, linear coupling constant J, temperature and external magnetic field. We confine ourselves in this paper to the case of $K < 0$ and $J < 0$ that is relevant to the recent experiment conducted on ^{23}Na atoms. Our study shows that there is thermal entanglement in the regime of $K < J$ when $K < 0$ and $J < 0$. Because some kinds of atoms satisfy the condition and others do not, the result shows what kinds of atoms can be used to produce thermal entanglement. For the multi-particle system, the thermal entanglement does not decrease greatly, and the critical temperatures decrease only slightly.

2. Properties of bosonic atom thermal entanglement for two atoms

Consider a lattice formed by three orthogonal laser beams with effective potential $V(x, y, z) = V_x \sin^2(kx) + V_y \sin^2(ky) + V_z \sin^2(kz)$, where k is the wave vector of the lasers, and the bosonic atoms with spin 1 are trapped in the wells. The strengths $V_{x,y,z}$ are proportional to the intensities of the laser beams

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and can be adjusted separately. We assume that only one orbital state is involved for each well, and we may use an effective Hamiltonian of the Bose–Hubbard form[14] to describe the system. The atoms in the Mott regime make sure that each well contains only one atom. In the case of $V_x \ll V_{y,z}$, the coupling along the y and z direction can be ignored. Thus the system is a collection of one-dimensional spin chains. The effective Hamiltonian of bosonic atoms can be written as[15]

$$H = \sum_i J(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + K(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2, \qquad (1)$$

where $J = -\frac{2t^2}{U_2}$, $K = -\frac{2t^2}{3U_2} - \frac{4t^2}{U_0}$ with t the hopping between adjacent sites. The hopping matrix elements depend on the direction of hopping, i.e. $t_{x,y,z} \sim (E_R V_{x,y,z})^{1/2} \mathrm{e}^{-2(V_{x,y,x}/E_R)^{1/2}}$, where $E_R \equiv \hbar^2 k^2/2m$ and m is the mass of the atoms. $U_s \sim (E_R V_x V_y V_z)^{1/4} k a_s$ $(s = 0, 2)$ represents the Hubbard repulsion potential with total spin s; the potential U_s with $s = 1$ is not allowed due to the identity of the bosons with one orbital state per well. $\boldsymbol{S}_i (i = 1, 2)$ denotes the spin vector $\boldsymbol{S}_i = (S_{ix}, S_{iy}, S_{iz})$. The sum is over the site labels. In the Mott regime, the hopping is very small, i.e. $t \to 0$, and atoms are trapped in the potential well, while non-linear spin interactions are in principle significant. This is the reason why we use the effective Hamiltonian Eq.(1). In fact, the Hamiltonian has another general form $H = \sum_i \cos\gamma(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + \sin\gamma(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2$ [15,16] in units of $\sqrt{J^2 + K^2}$, where $0 < \gamma < 2\pi$ for general cases. For $\gamma = 0$ and π this reduces to the usual Heisenberg antiferromagnet and ferromagnet. But the Hamiltonian in Eq.(1) differs from the usual Heisenberg model by the nonlinear couplings for J and $K < 0$. We will consider $\pi < \gamma < \frac{3\pi}{2}$($J$ and $K < 0$). When the magnetic field $\boldsymbol{B}$ along the z-direction is assumed, the Hamiltonian Eq.(1) becomes

$$\begin{aligned} H =& \sum_i J(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + K(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \\ &+ B(S_{iz} + S_{i+1z}), \end{aligned} \qquad (2)$$

where $S_i = 1 (i = 1, 2)$ is the total spin for each site and its components take the form

$$S_{ix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$S_{iy} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix},$$

$$S_{iz} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad (3)$$

We first consider the simple case that the system consists of two atoms. To get the thermal entanglement, we present the eigenvalues and the corresponding eigenstates of the Hamiltonian Eq.(2),

$$\begin{aligned} E_1 =& K + J - B, \ |\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0,-1\rangle + |-1,0\rangle); \\ E_2 =& K + J + B, \ |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|1,0\rangle + |0,1\rangle); \\ E_3 =& K - J + B, \ |\Psi_3\rangle = \frac{1}{\sqrt{2}}(-|1,0\rangle + |0,1\rangle); \\ E_4 =& K - J - B, \ |\Psi_4\rangle = \frac{1}{\sqrt{2}}(-|0,-1\rangle + |-1,0\rangle); \\ E_5 =& K + J, \ |\Psi_5\rangle = \frac{1}{\sqrt{6}}(|1,-1\rangle + |-1,1\rangle + 2|0,0\rangle); \\ E_6 =& K - J, \ |\Psi_6\rangle = \frac{1}{\sqrt{2}}(|1,-1\rangle - |-1,1\rangle); \\ E_7 =& K + J + 2B, \ |\Psi_7\rangle = |1,1\rangle; \\ E_8 =& K + J - 2B, \ |\Psi_8\rangle = |-1,-1\rangle; \\ E_9 =& 4K - 2J, \ |\Psi_9\rangle = \frac{1}{\sqrt{3}}(|1,-1\rangle + |-1,1\rangle - |0,0\rangle). \end{aligned} \qquad (4)$$

The state of the system at thermal equilibrium is $\rho = Z^{-1}\exp(-\beta H)$, where $Z = \mathrm{Tr}[\exp(-\beta H]$ is the partition function and $\beta = 1/k_{\mathrm{B}}T$ (k_{B} is Boltzmann's constant. For simplicity, we will set $k_{\mathrm{B}} = 1$ hereafter). On the base of the eigenstates, the state of the system can be expressed as

$$\rho = \frac{1}{Z} \sum_{i=1}^{9} \mathrm{e}^{-\beta E_i} |\Psi_i\rangle\langle\Psi_i| \qquad (5)$$

with the partition function

$$\begin{aligned} Z =& 2\mathrm{e}^{-\beta K}\cosh(\beta B)(1 + 2\cosh(\beta J)) \\ &+ 2\mathrm{e}^{-\beta(K+J)}\cosh(2\beta B) + \mathrm{e}^{-\beta(4K-2J)}. \end{aligned} \qquad (6)$$

We will choose the negativity as the entanglement measure,[17]

$$E(\rho) = \frac{\left\|\rho^{T_A}\right\| - 1}{2}, \qquad (7)$$

where $\left\|\rho^{T_A}\right\|$ denotes the trace norm of the partial transpose ρ^{T_A}. As the negativity is a computable measure of entanglement for a bipartite system of any dimension, here we choose it to measure the thermal

entanglement. The negativity $E(\rho)$ is equivalent to the absolute value of the sum of the negative eigenvalues of ρ^{T_A},[18] i.e. $E(\rho) = \frac{\|\rho^{T_A}\| - 1}{2} = -\sum_i \lambda_i$, where λ_i is the negative eigenvalue. From the definition $E(\rho) = -\sum_i \lambda_i$, one can see that the maximum value of the absolute value of the sum of the negative eigenvalues of ρ^{T_A} may be greater than 1 as long as $\mathrm{tr}(\rho^{T_A}) = 1$. For different dimensions of density matrix, the maximum value is different. For example, the maximum is 0.5 for the two qubits with two levels. So, to make the maximum value equal to 1, in Ref.[19] the author uses the expression $\varepsilon(\rho) = -2\sum_i \lambda_i$ to measure an entangled two-body spin-1/2 state. In our case of spin 1 particles, because in the states $|\Psi_{1,2,3,4,6}\rangle$ the negativity is 0.5, in the states $|\Psi_5\rangle$ the negativity is smaller than 1 and the negativity is 1 in $|\Psi_9\rangle$, so the negativity value of the statistical mixture of these states is equal to or less than 1. In our plots, the value of 1 means maximal entanglement.

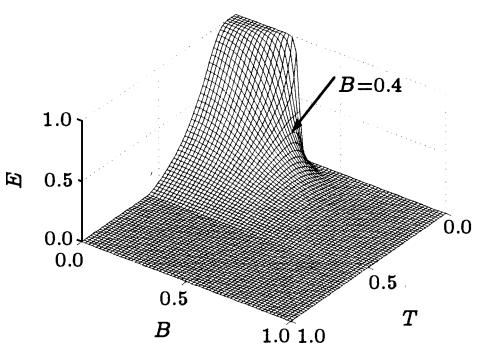


Fig.1. Thermal entanglement between two bosonic atoms vs the external magnetic field B and temperature T, where $K = -0.3$, $J = -0.065$.

Because the density matrix is nine dimensional, it is very tedious to write the eigenvalues of ρ^{T_A}. Here, we directly plot the entanglement as different functions of the parameters. According to condensing theory, for sodium in the Mott regime when a red detuned laser is involved, the hopping matrix element between adjacent sites $t = 0.07E_{\mathrm{R}}$, $U_2 = 0.15E_{\mathrm{R}}$, thus $J = -0.065E_{\mathrm{R}}$ (hereafter J and K are all in units of E_{R}). We will fix the value of J and let nonlinear couplings K and external magnetic field B change. We have performed extensive numerical calculations for two atoms; some selected results are presented in Fig.1 and Fig.2. Figure 1 shows the plot of the negativity as a function of the magnetic field B and temperature T. For $B = 0$, the state $|\Psi_9\rangle$ is the ground state, and the others in Eq.(4) are excited states. In this case, the maximal entanglement is at $T = 0$ and it decreases with the increasing of T due to mixing of the excited states with the ground state. At $T = 0$, with the increasing of B, the entanglement vanishes suddenly as B passes the critical value $B_{\mathrm{c}} = 3/2(J - K)$, which corresponds to a quantum phase transition. In this case the state $|\Psi_8\rangle$ becomes the ground state, so no entanglement exists. However, for a higher B, we may increase the entanglement by increasing T, for example $B = 0.4$ (Fig.1). That is to bring entangled eigenstates such as $|\Psi_i\rangle$ $(i = 1, 2, 3, 4, 5, 6, 9)$ into mixing with the ground state. It is interesting to note that the critical field B_{c} depends on both the linear coupling J and the nonlinear coupling K. With $K = 0$, B_{c} gives rise to $3/2J$ that means no entanglement in this case (i.e. for $J < 0$) at any temperature and with any value of magnetic field. We can see that J, K and B together determine the ground state properties of the system instead of J and B in the isotropic Heisenberg model. This is a quite different feature from previous studies. For example, we may choose K, J and B such that $|\Psi_1\rangle$ is the ground state of the system; it is an entangled state but not a maximally entangled one.

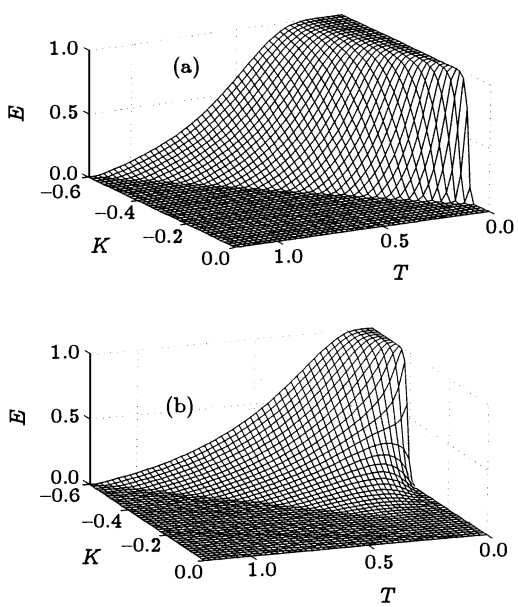


Fig.2. Thermal entanglement as a function of the temperature T and the nonlinear coupling constant K with a specific $J = -0.065$. (a) $B = 0$ and (b) $B = 0.5$.

For $B = 0$ and $J = -0.065$, thermal entanglement exists only when $|K| > 0.1$, as Fig.2(a) shows. This indicates that $|K| > |J|$ is a necessary condition for thermal entanglement to exist. The result is still true for many-body systems which we will show in the next section.

To understand the condition, we deduce the analytic condition under which the entanglement exists.

When $B = 0$, the eigenvalues of Eq.(4) are degenerate, and the nine eigenvalues becoming three, which makes it possible to obtain the eigenvalues of ρ^{T_A}. The three eigenvalues of ρ^{T_A} are

$$\begin{aligned}\lambda_1 =&\frac{1}{z}\left[\frac{1}{3}\mathrm{e}^{-\beta(4k-2j)} + \frac{2}{3}\mathrm{e}^{-\beta(j+k)}\right.\\ &\left. + \mathrm{e}^{-\beta k}(\mathrm{e}^{-\beta j} - \mathrm{e}^{\beta j})\right],\\ \lambda_2 =&\frac{1}{6z}\mathrm{e}^{-\beta(4k-2j)}[3\mathrm{e}^{\beta(3k-j)} + 2 + \mathrm{e}^{\beta(3k-3j)}],\\ \lambda_3 =&\frac{1}{6z}\mathrm{e}^{-\beta(4k-2j)}[3\mathrm{e}^{\beta(3k-j)} - 2 + 5\mathrm{e}^{\beta(3k-3j)}]. \quad (8)\end{aligned}$$

Here, we discuss the case $j < 0$ so that $\lambda_1 > 0$ and $\lambda_2 > 0$. Therefore, the ρ^{T_A} has negative eigenvalue only when $\lambda_3 < 0$. Thus, for $B = 0$ and with a specific temperature T, the condition for thermal entanglement to exist is

$$K < J - \frac{T}{3}\ln\left(\frac{5 + 3\mathrm{e}^{2J/T}}{2}\right), \quad (9)$$

which apparently shows the condition $|K| > |J|$. We call this value of K the critical nonlinear coupling which means that when $|K|$ is smaller than the critical value the entanglement disappears. On the other hand, it gives rise to an analytical expression for the critical temperature T_c in the case of $B = 0$ when $T > 0$ by solving $K = J - \frac{T_c}{3}\ln\left(\frac{5 + 3\mathrm{e}^{2J/T_c}}{2}\right)$, which shows again that T_c depends on J and K. In addition to the above feature, there is evidence that the critical temperature is a monotonic function of $|K|$ with $B = 0$: the larger the nonlinear coupling constant $|K|$, the higher the critical temperature. This point is changed when the magnetic field is present (Fig.2(b)): one sees the threshold temperature is no longer a monotonic function of $|K|$, but the condition $|K| > |J|$ is still true for $B \neq 0$.

Though we let the coupling K change, in the Mott regime, the value of the tunnelling and the interaction are fixed for a kind of atoms. However, there are different kinds of atoms in which the relations between J and K are different. For example, for ^{23}Na atoms, $U_2 > U_0$; hence $K < J < 0$, while for ^{87}Rb, $U_0 > U_2$, in this case, $J < K < 0$. Therefore, the condition $|K| > |J|$ (for $B = 0$) can tell us in which kinds of atoms the entanglement exists.

3. Thermal entanglement for 1D many-body bosonic atoms chain

In experiment, bosonic atoms with spin 1 in optical lattices exist in the form of multi-particle chains. As a possible entanglement resource, we need to make clear the structure of multi-particle entanglement. Now, we study the multi-particle entanglement in 1D optical lattice chains.

Figure 3 shows the change of nearest neighbor entanglement with T in N-qubit case for $B = 0$. As in the 1D isotropic Heisenberg chain,[6] we can also observe that the curves for entanglement in the case of even and odd N approach each other as N increases, although we do not plot the merger due to the complexity of the high dimension 3^N density matrix. The critical temperature for $N = 3$ differs little from that for $N = 4$, and the curves for $N = 6$ and $N = 7$ almost merge in the relatively higher temperature region. This is easily understood, because for very large N, there is no difference for the nearest neighbour entanglement whether we add or subtract a qubit. Comparing Fig.3 (large N) with Fig.1 ($N = 2$), we see that the maximum for large N decreases a lot but the critical temperatures are almost the same.

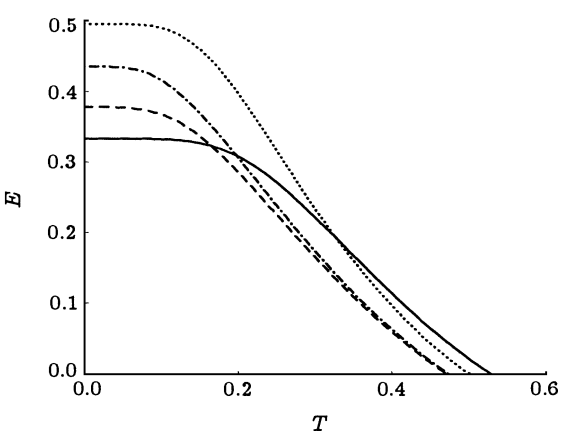


Fig.3. Nearest neighbour thermal entanglement vs temperature T for several values of atomic number N (from top to bottom at $T = 0$, $N = 4$,$N = 6$,$N = 7$ and N=3). The parameters chosen are $J = -0.065$, $K = -0.3$ in units of E_R and $B = 0$.

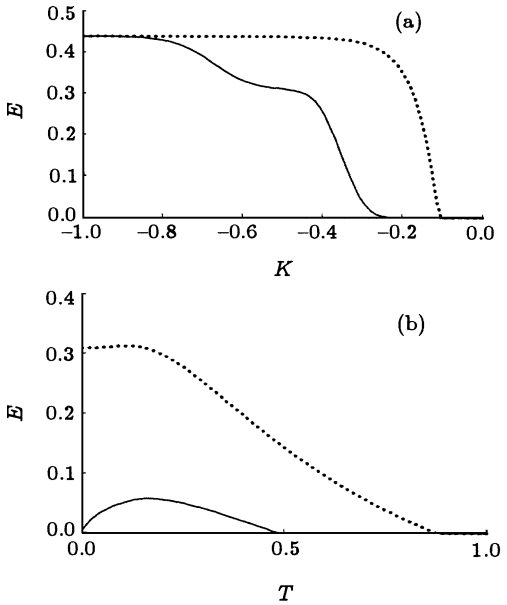


Fig.4. Thermal entanglement vs nonlinear cooling K or temperature T, where (a) dotted line for $B = 0$ and solid line for $B = 0.3$; (b) dotted line $K = -0.5$ and solid line $K = -0.3$. For all plots $J = -0.065$, $N = 6$.

For $N = 6$, Fig.4 shows the thermal entanglement changing with K or T. To the multi-particle system, the nonlinear coupling is still the key factor to produce the thermal entanglement, which is shown in Fig.4(a). Only when the nonlinear couplings are larger than a certain critical value does the entanglement exist. For $B = 0$ the critical value of K is smaller than that for $B = 0.3$. The introduction of magnetic field makes the critical nonlinear coupling increase. We also notice that the absolute values of critical coupling in both cases in Fig.4(a) are larger than $|J|$: that is, the condition of thermal entanglement existing, $|K| > |J|$, is still true for multi-particles no matter whether $B = 0$ or $B \neq 0$. Figure 4(b) shows the nearest neighbour entanglement changing with temperature when $B = 0.3$. For $K = -0.3$, the thermal entanglement can be increased by increasing temperature; this is because in this case the value of $B = 0.3$ is already larger than the critical magnetic field. But for $K = -0.5$, the value of $B = 0.3$ is smaller than the corresponding critical magnetic field, so we cannot observe the phenomenon. Comparing the curve for $N = 6$ in Fig.3 with the curve for $K = -0.3$ in Fig.4(b), we can say that the magnetic field being larger than the critical magnetic field is the condition of existence of the phenomenon in which entanglement can be increased through increasing temperature, which is similar to the isotropic antiferromagnetic Heisenberg model.[6]

For the multi-particle system, the thermal entanglement does not decrease greatly, and the critical temperatures only decreases slightly. But the condition for the thermal entanglement to exist, $|K| > |J|$, is still true for the multi-particle system. When the magnetic field is greater than the critical magnetic field, one can also observe that the thermal entanglement can be increased through increasing the temperature.

4. Discussion and conclusion

To conclude, we have studied the thermal entanglement in an optical lattice. The dependence of the thermal entanglement measured by the negativity on the linear coupling constant J, the nonlinear coupling constant K, the external magnetic field and temperature was presented and discussed. There are two points different from the isotropic Heisenberg model: one is the dimensionality and the other is the nonlinearity of the couplings. For J $(K) < 0$, the condition of thermal entanglement existing, $K < J < 0$, holds for either the two-particle or the multi-particle system. For the multi-particle system, the thermal entanglement does not decrease greatly, and the critical temperatures only decreases slightly.

Our study indicates that only when $|K| > |J|$ can entanglement exist if $J(K) < 0$. This result is important in that it shows what kind of atoms can be used to prepare thermal entanglement. We may choose ^{23}Na trapped in an optical lattice as the system. Because for ^{23}Na atoms $U_2 > U_0 > 0$,[15,20] hence $K < J < 0$. So the ^{23}Na atoms satisfy the condition of thermal entanglement $|K| > |J|$. The ^{23}Na atoms have hyperfine spin 1, and the interaction among them is antiferromagnetic, which is essential for the thermal entanglement to exist in isotropic media, similar to the results for the isotropic Heisenberg model.[6]

References

[1] Bennett C H and DiVincenzo D P 2000 *Nature* **404** 247
[2] Nielsen M A and Chuang I L 2000 *Quantum computation and quantum information* (Cambridge: Cambridge University Press)
[3] Raimond J M *et al* 2001 *Rev. Mod. Phys.* **73** 565
Vedral V 2002 *Rev. Mod. Phys.* **74** 197
[4] Zheng S B 2003 *Chin. Phys.* **12** 977
[5] Liu X J and Fang M F 2003 *Chin. Phys.* **12** 971
[6] Arnesen M C, Bose S and Vedral V 2001 *Phys. Rev. Lett.* **87** 017901

[7] OConnor K M and Wootters W K 2001 *Phys. Rev.* A **63** 052302
[8] Wang X 2002 *Phys. Rev.* A **66** 044305
Wang X 2002 *Phys. Rev.* A **66** 034302
[9] Wang X 2001 *Phys. Rev.* A **64** 012313
Kamta G L and Starace A F 2002 *Phys. Rev. Lett.* **88** 107901
Zhou L, Song H S, Guo Y Q and Li C 2003 *Phys. Rev.* A **68** 024301
[10] Gunlycke D, Kendon V M, Vedral V and Bose S 2001 *Phys. Rev.* A **64** 042302
[11] Nielsen M A e-print quant-ph/0011036
[12] Mancini S and Bose S 2004 *Phys. Rev.* A **70** 022307
[13] Birkl G *et al* 2001 *Optics Commun.* **191** 67
[14] Jaksch D, Bruder C, Cirac J I, Gardiner C W and Zoller P 1998 *Phys. Rev. Lett.* **81** 3108
[15] Yip S K 2003 *Phys. Rev. Lett.* **90** 250402
[16] Chubukov A V 1991 *Phys. Rev.* B **43** 3337
[17] Vidal G and Werner R F 2002 *Phys. Rev.* A **65** 032314
[18] Zyczkowski K *et al* 1998 *Phys. Rev.* A **58** 883
[19] Lee Jinhyoung and Kim M S 2000 *Phys. Rev. Lett.* **84** 4236
[20] Stenger J *et al* 1999 *Nature* **396** 245