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Fractional Fourier transform of flat-topped multi-Gaussian beams based on the Wigner distribution function*

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By using the Wigner distribution function, the fractional Fourier transform (FRFT) of flat-topped multi-Gaussian (FTMG) beams is studied. Analytical expressions for the intensity distribution, beam width, far-field divergence angle, M^2 factor and K parameter of FTMG beams are derived. The influence of fractional order on transformation properties of FTMG beams in the FRFT plane is illustrated with numerical examples.

Keywords: Wigner distribution function, flat-topped multi-Gaussian beam, fraction Fourier transform

PACC: 4210, 4230K

1. Introduction

The fractional Fourier transform (FRFT) was originally introduced in 1980 by Namias.[1] Mendlovic, Ozaktas and Lohmann introduced it into optics and gave its optical implementation.[2−5] The FRFT is a generalization of the conventional Fourier transform, and has been extensively applied to the optical information processing, beam propagation and transformation, etc.[6−10]

On the other hand, flat-topped beams are of practical interest, and there are several models, such as the super-Gaussian beam, flatted Gaussian beam and flat-topped multi-Gaussian (FTMG) beam, etc[11−13] proposed to characterize propagation properties of flat-topped beams. The aim of the present paper is to study the FRFT of FTMG beams. In Section 1 analytical expressions for the intensity distribution in the FRFT plane, beam width, far-field divergence angle, M^2 factor and kurtosis parameter (K parameter) of FTMG beams are derived and analysed by using the Wigner distribution function (WDF). Numerical calculation results and discussions are presented in Section 3. Finally, Section 4 summarizes the main results obtained in this paper.

2. Analytical expressions for the FRFT of FTMG beams

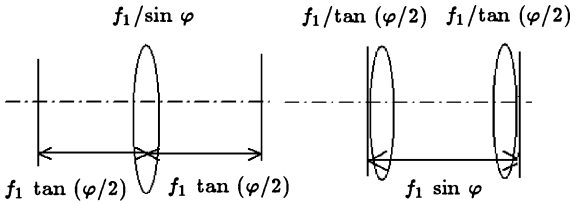
It is well-known[5] that the FRFT with fractional order p corresponds to the WDF rotated by an angle of

$$\varphi = \frac{\pi}{2}p. \tag{1}$$

Two types of optical systems that are often used to perform the FRFT on the basis of the rotation of the WDF are shown in Fig.1. Type I is composed of a lens of focal length $\dfrac{f_1}{\sin\varphi}$, with two equal distances $f_1 \tan\dfrac{\varphi}{2}$ on opposite sides of the lens. Type II consists of two lenses of focal length $\dfrac{f_1}{\tan\dfrac{\varphi}{2}}$ separated by a distance $f_1 \sin\varphi$, where f_1 is a constant.

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(a) Type I. (b) Type II.

Fig.1. Optical systems for performing the FRFT (a) Type I; (b) Type II.

Assume that the field of FTMG beams combined with N individual Gaussian beamlets with the equal waist width w_0 and equal separation x_d in the incident plane ($z=0$) is expressed as[13]

$$E(x,z=0)=\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left[-\left(\frac{x-nx_d}{w_0}\right)^2\right]. \quad (2)$$

The cross-spectral density function of FTMG beams in the incident plane ($z=0$) is given by[14]

$$\begin{aligned}W(x_1,x_2)=&E(x_1,z=0)\cdot E^*(x_2,z=0)\\=&\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left[-\left(\frac{x_1-mx_d}{w_0}\right)^2\right]\\&\times\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left[-\left(\frac{x_2-nx_d}{w_0}\right)^2\right],\end{aligned} \quad (3)$$

where the star * denotes the complex conjugate.

From Eq.(3) the WDF of FTMG beams in the incident plane turns out to be[5]

$$\begin{aligned}&h_0(x_0,u_0)\\=&\iint_{\infty}W\left(x_0-\frac{1}{2}x',x_0+\frac{1}{2}x'\right)\exp\left[-\frac{2\pi\mathrm{i}}{\lambda f_1}u_0x'\right]\mathrm{d}x'\\=&\sqrt{2\pi}w_0\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left\{-\frac{1}{2}\left(\frac{2\pi}{\lambda f_1}w_0u_0\right)^2\right.\\&\left.-\frac{[2x_0-(m+n)x_d]^2}{2w_0^2}\right\},\end{aligned} \quad (4)$$

where u_0 is the spatial frequency, and λ is the wavelength related to the wave number k by $\lambda=\frac{2\pi}{k}$.

The propagation of the WDF obeys the law[15]

$$h_p(x,u)=h_0(x\cos\varphi-u\sin\varphi,u\cos\varphi+x\sin\varphi). \quad (5)$$

Substituting Eqs.(1) and (4) into Eq.(5), we obtain the WDF of FTMG beams in the FRFT plane in the form

$$\begin{aligned}h_p(x,u)=&\sqrt{2\pi}w_0\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left\{-\frac{1}{2}\left[\frac{2\pi}{\lambda f_1}w_0\right.\right.\\&\left.\times\left(u\cos\frac{p\pi}{2}+x\sin\frac{p\pi}{2}\right)\right]^2\\&\left.-\frac{\left[2\left(x\cos\frac{p\pi}{2}-u\sin\frac{p\pi}{2}\right)-(m+n)x_d\right]^2}{2w_0^2}\right\}.\end{aligned} \quad (6)$$

The intensity distribution of the FTMG beams in the FRFT plane with the fractional order being p is obtained readily from Eq.(6) as

$$\begin{aligned}I(x,p)=&\int h_p(x,u)\mathrm{d}u=\frac{2\pi w_0}{\sqrt{g}}\\&\times\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left\{-\frac{1}{g}\left(\frac{2\pi w_0}{\lambda f_1}\right)^2\right.\\&\times\left[2x^2-2(m+n)x_dx\cos\frac{p\pi}{2}\right.\\&\left.\left.+(m^2+n^2)x_d^2\cos^2\frac{p\pi}{2}\right]\right\},\end{aligned} \quad (7)$$

where $g=4+2\left[\left(\frac{\pi w_0^2}{\lambda f_1}\right)^2-1\right](\cos p\pi+1)$. Equation (7) indicates that the intensity distribution of FTMG beams in the FRFT plane depends on the fractional order p, beam number N, waist width w_0 and separation x_d.

By letting $x=0$, the on-axis intensity of FTMG beams is expressed as

$$\begin{aligned}I(0,p)=&\frac{2\pi w_0}{\sqrt{g}}\\&\times\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\exp\left\{-\frac{1}{2g}\left(\frac{2\pi w_0}{\lambda f_1}\right)^2\right.\\&\left.\times(m^2+n^2)x_d^2(\cos p\pi+1)\right\}.\end{aligned} \quad (8)$$

The second-moments $\langle x^2\rangle$, $\langle u^2\rangle$ and $\langle xu\rangle$ are defined as[16]

$$\langle x^2\rangle=\frac{\int_{-\infty}^{\infty}x^2h_p(x,u)\mathrm{d}x\mathrm{d}u}{\int_{-\infty}^{\infty}h_p(x,u)\mathrm{d}x\mathrm{d}u},$$

$$\langle u^2\rangle = \frac{\int_{-\infty}^{\infty} u^2 h_p(x,u)\,\mathrm{d}x\mathrm{d}u}{\int_{-\infty}^{\infty} h_p(x,u)\,\mathrm{d}x\mathrm{d}u}, \qquad \langle xu\rangle = \frac{\int_{-\infty}^{\infty} xu h_p(x,u)\,\mathrm{d}x\mathrm{d}u}{\int_{-\infty}^{\infty} h_p(x,u)\,\mathrm{d}x\mathrm{d}u}. \tag{9}$$

From Eqs.(6) and (9), the beam width $w(p)$, far-field divergence angle $\theta(p)$ and the factor M^2 are given by[17]

$$w(p) = 2\sqrt{\langle x^2\rangle} = 2\sqrt{\left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} r_{mn}\right] \Big/ \left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} p_{mn}\right]}, \tag{10}$$

$$\theta(p) = 2\sqrt{\langle u^2\rangle} = 2\sqrt{\left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} q_{mn}\right] \Big/ \left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} p_{mn}\right]}, \tag{11}$$

$$\begin{aligned} M^2 =& 2k\sqrt{\langle x^2\rangle\langle u^2\rangle - \langle xu\rangle^2} \\ =& \left[\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m'=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n'=-\frac{N-1}{2}}^{\frac{N-1}{2}} p_{mn}p_{m'n'}F_{1mn}F_{2m'n'}\right]^{1/2} \Big/ \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} p_{mn}, \end{aligned} \tag{12}$$

where

$$r_{mn} = p_{mn}\left[\left(\frac{F_{1mn}}{8\left(\frac{\pi w_0}{\lambda f_1}\right)^2} + \frac{w_0^2 F_{2mn}}{8}\right) - \left(\frac{F_{1mn}}{8\left(\frac{\pi w_0}{\lambda f_1}\right)^2} - \frac{w_0^2 F_{2mn}}{8}\right)\cos p\pi\right], \tag{13}$$

$$q_{mn} = p_{mn}\left[\left(\frac{F_{1mn}}{8\left(\frac{\pi w_0}{\lambda f_1}\right)^2} + \frac{w_0^2 F_{2mn}}{8}\right) + \left(\frac{F_{1mn}}{8\left(\frac{\pi w_0}{\lambda f_1}\right)^2} - \frac{w_0^2 F_{2mn}}{8}\right)\cos p\pi\right], \tag{14}$$

$$p_{mn} = \exp\left[-\frac{(m-n)^2\left(\frac{x_d}{w_0}\right)^2}{2}\right], \tag{15}$$

$$F_{1mn} = 1 - (m-n)^2\left(\frac{x_d}{w_0}\right)^2, \tag{16}$$

$$F_{2\alpha\beta} = 1 + (\alpha+\beta)^2\left(\frac{x_d}{w_0}\right)^2, \quad (\alpha = m, \beta = n, \text{ or } \alpha = m', \beta = n'). \tag{17}$$

Equations (10), (11) and (12) indicate that the factor M^2 of the FTMG beams is dependent on the beam number N, waist width w_0 and separation x_d, but independent of the fractional order p. Equation (12) is essentially consistent with Eq.(13) in Ref.[18].

The kurtosis parameter (K parameter) of the FTMG beams is defined as[19]

$$K = \frac{\langle x^4\rangle}{\langle x^2\rangle^2}, \tag{18}$$

where

$$\langle x^4\rangle = \frac{\int_{-\infty}^{\infty} x^4 h(x,u)\,\mathrm{d}x\mathrm{d}u}{\int_{-\infty}^{\infty} h(x,u)\,\mathrm{d}x\mathrm{d}u}. \tag{19}$$

From Eqs.(11) and (18) the K parameter turns out to be

$$K(p)=\left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}p_{mn}\right]\cdot\left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}S_{mn}\right]\Bigg/\left[\sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}}\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}}r_{mn}\right]^2, \tag{20}$$

where

$$\begin{aligned}S_{mn}=&\frac{p_{mn}}{16\left(\frac{2\pi w_0}{\lambda f_1}\right)^4}\left\{4\left(1-\cos p\pi\right)^2\left[3-6\left(m-n\right)^2\left(\frac{x_d}{w_0}\right)^2+\left(m-n\right)^4\left(\frac{x_d}{w_0}\right)^4\right]\right.\\&+4\left(\frac{\pi w_0^2}{\lambda f_1}\right)^4\left(1+\cos p\pi\right)^2\left[3+6\left(m+n\right)^2\left(\frac{x_d}{w_0}\right)^2+\left(m+n\right)^4\left(\frac{x_d}{w_0}\right)^4\right]\\&\left.+24\left(\frac{\pi w_0\sin p\pi}{\lambda f_1}\right)\left[1+4mn\left(\frac{x_d}{w_0}\right)^2-\left(m^2-n^2\right)^2\left(\frac{x_d}{w_0}\right)^4\right]\right\}.\end{aligned} \tag{21}$$

Equations (12) and (18) show that the K parameter of the FTMG beams depends on the fractional order p, beam number N, waist width w_0 and separation x_d.

3. Results of numerical calculation and discussion

Numerical calculations were performed using the analytical expressions derived in the above section to illustrate the transformation properties of the FTMG beams in the FRFT plane. Figure 2 illustrates the intensity distributions of a FTMG beam with $N = 9$ in the FRFT plane for different values of the fractional order. Other calculation parameters are w_0=0.07mm, x_d=0.075mm, and $\lambda f_1 = 1$. From Fig.2 we see that the fractional order affects the intensity profile. For $p = 0, 2$, there is a flat-topped profile, and the period $T = 2$; whereas for $p = 0.05$ and 1, the intensity has a central dip and a bell-shape profile, respectively.

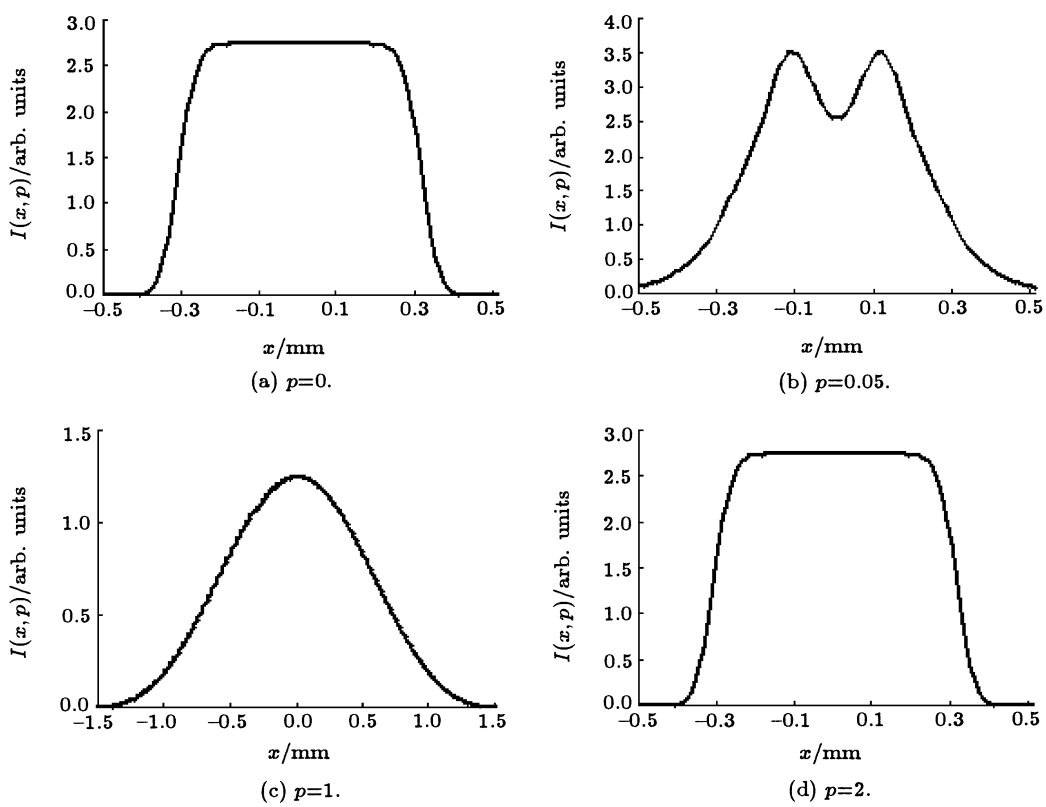


Fig.2. The intensity distributions of FTMG beams with $N = 9$ in the FRFT plane for different values of the fractional order (a) $p = 0$; (b) $p = 0.05$; (c) $p = 1$; (d) $p = 2$. The calculation parameters are given in the text.

The on-axis intensity distribution of the FTMG beams as a function of fractional order p for different values of the beam number N and separation x_d are plotted in Figs.3 (x_d=0.075mm) and 4(N=9), respectively. Other calculation parameters are w_0=0.07mm and λf_1=1. As can be seen, the $I(0,p)$-p curves vary with a period $T=2$, and the on-axis intensity distributions depend on the beam number N and separation x_d.

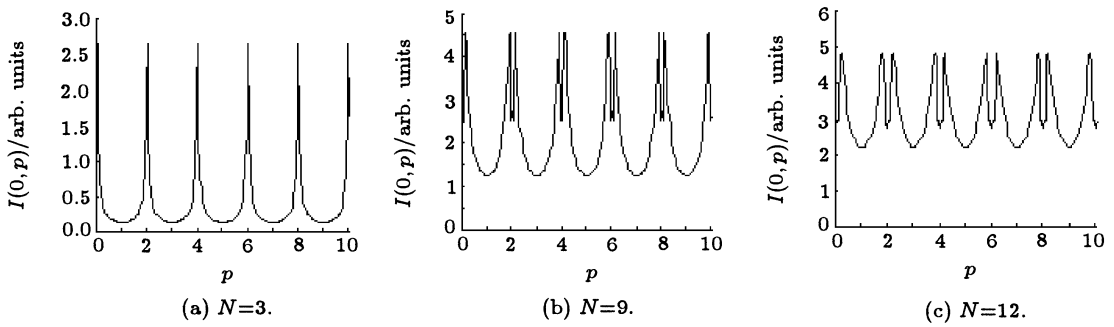


Fig.3. The on-axis intensity distributions of FTMG beams with x_d=0.075mm as a function of fractional order p for different values of the beam number (a) $N=3$; (b) N=9; (c) $N=12$.

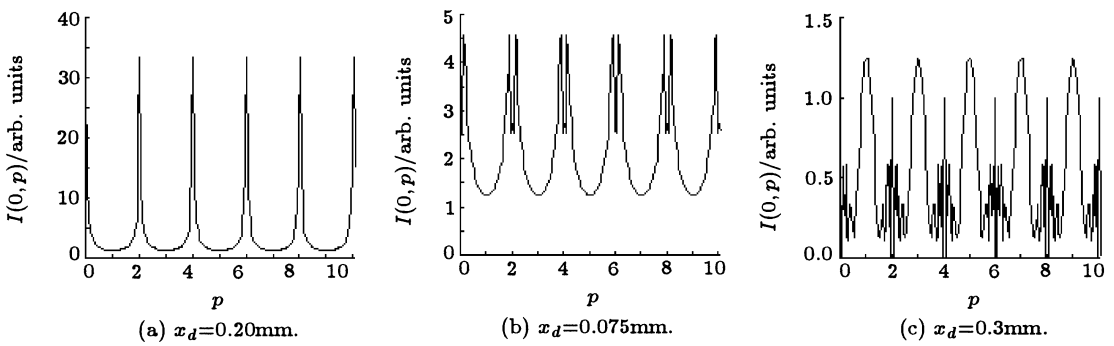


Fig.4. The on-axis intensity of FTMG beams with N=9 as a function of fractional order p for different values of the separation (a) x_d=0.02mm; (b) x_d=0.075mm; (c) x_d=0.3mm.

The beam width versus fractional order p for different values of separation x_d is depicted in Fig.5 ($N=9$) and the other calculation parameters are w_0=0.07mm and λf_1=1. From Fig.5 it follows that the variation of the beam width with fractional order p is periodic, with T=2. For a given p the beam width increases with increasing x_d.

The variation of the far-field divergence angle versus fractional order p is given in Fig.6. The calculation parameters are the same as those in Fig.5. We see that the far-field divergence angle varies with p in a similar way as the beam width does, but for $p=0,2,4,...$, where the maximum $\theta(p)$ is achieved, we have the minimum $w(p)$, and vice versa for $p=1,3,5,....$

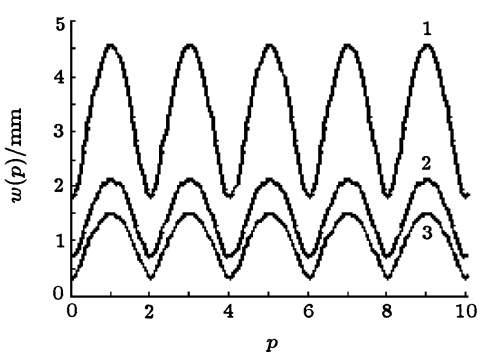


Fig.5. The beam width versus fractional order p for different values of the separation x_d (1. x_d=0.3mm; 2. x_d=0.15mm; 3. x_d=0.075mm).

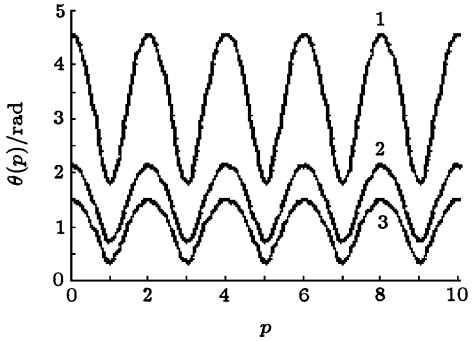


Fig.6. The far-field divergence angle versus fractional order p for different values of the separation x_d (1. x_d=0.3mm; 2. x_d=0.15mm; 3. x_d=0.075mm).

4. Conclusion

In this paper, the transformation properties of FTMG beams in the FRFT plane have been studied in detail. Based on the WDF, analytical expressions for the intensity distribution, beam width, far-field divergence angle, the factor M^2 and the K parameter of FTMG beams have been derived and illustrated with numerical examples. It has been found that the intensity in the FRFT plane depends on the fractional order p, beam number N, waist width w_0 and separation x_d. The factor M^2 is independent of the fractional order p, whereas the K parameter is dependent on p. The variation of the intensity, beam width, far-field divergence angle and K parameter with fractional order p is periodic, and the period is $T = 2$. In addition, our results have shown that the WDF provides a useful and simple tool in studying the FRFT of beams, because by means of the WDF the intensity and beam characteristic parameters can be readily expressed in a closed-form.

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