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Coherence resonance and synchronization of Hindmarsh–Rose neurons with noise*

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Noise effects on coherence resonance and synchronization of Hindmarsh–Rose (HR) neuron model are studied. The coherence resonance of a single HR neuron with Gaussian white noise added to the membrane potential is investigated in situations before, near and after the Hopf bifurcation, separately, with the external direct current as a bifurcation parameter. It is shown that even though there is no coupling between neurons, uncoupled identical HR neurons driven by a common noise can achieve complete synchronization when the noise intensity is higher than a critical value. Furthermore, noise also enhances complete synchronization of weakly coupled neurons. It is concluded that synchronization in bursting neurons is easier to be induced than in spiking ones, and coupling enhances the sensitivity of synchronization of neurons to noise stimulus.

Keywords: coherence resonance, synchronization, Hindmarsh–Rose neuron, Gaussian white noise
PACC: 0545, 4350, 8725

1. Introduction

Nowadays, two kinds of phenomena in nonlinear systems, synchronization[1−3] and noise effects,[4,5] have attracted considerable research interest in different fields of natural sciences and engineering. From the most general point of view, synchronization is understood as an adjustment of some relations among state variables, frequencies or phases of two or more dynamical systems during their interactions. With the development of nonlinear dynamics, the classical concept of synchronization has been extended to the case of chaotic oscillators. In recently published papers, many kinds of chaos synchronizations are well described: namely, complete synchronization (CS), lag synchronization, phase-frequency synchronization, and generalized synchronization.

Noise exists widely in natural systems, such as those in physics, chemistry, biology, economy and engineering. Constructive effects of noise in nonlinear systems have been investigated extensively in the context of stochastic resonance (SR)[6] and coherence resonance (CR).[7] Unlike SR, which acts as the associated effect induced by noise and weak external signals, CR is mainly observed in excitable systems, in which pure noise is sufficient to generate most of the coherent motions without external signals. A bistable or excitable system can generate a coherent response to a subthreshold external forcing in the presence of a suitable amount of noise. The coherence is maximal at a certain optimal noise intensity.

In neural systems, noise arises from many different sources, such as the quasi-random release of neurotransmitter by the synapse, the random switching of ion channels, and most important random synaptic input from other neurons. Noise was traditionally seen as a disturbance which limits the accuracy of information transfer, while it is true that intrinsic noise presented in excitable cells can play a relevant role in the detection and processing of signals. In recent years, noise sources are viewed as important dynamical components, and there has been a great deal of interest in the effect of noise on the dynamics of excitable cell systems, which can stimulate distinct firing activities of excitable cells.[7−12] CR can be found in neuron models including the FitzHugh–Nagumo model, the Hindmarsh–Rose (HR) model, the Morris–Lecar model, the Hodgkin–Huxley and Chay

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models, with various noises, such as Gaussian white noise (GWN), Ornstein–Uhlenbeck noise (OUN) and quasi-monochromatic noise (QMN).

Noise may influence synchronization in different ways.[13] It was shown that noise can induce synchronization in chaotic oscillators,[14−16] as well as in periodic oscillators.[17] Identical systems, which are not coupled but subjected to a common noise, may achieve CS at a large enough noise intensity. Meanwhile, noise can induce weaker degrees of synchronization, especially phase synchronization, when two systems are nonidentical or the noise intensity is not strong enough to induce CS. For weakly coupled nonidentical chaotic oscillators, noise can enhance phase synchronization when the coupling is not strong enough to reach phase synchronization.

In this paper, we mainly consider the noise effects in two HR neurons with a common GWN. We firstly investigate the CR of a single HR neuron subjected to GWN, whenever it experiences quiescent state, subthreshold oscillating and periodic firing, separately, which correspond to the situations before, near and after the Hopf bifurcation. The external direct current added to a neuron is treated as a bifurcation parameter. This is different from the other study in which only the noise effect on subthreshold oscillating neurons is discussed. Based on these results, we further discuss the noise influence on synchronization, including noise-induced CS of two HR neurons driven by a common GWN and noise-enhanced CS of two weakly coupled HR neurons. Here we investigate the noise effects on synchronization not only for chaotic firing neurons, but also for periodic firing and subthreshold oscillating ones.

An outline of this paper is given as follows. The deterministic and stochastic HR neuron models are introduced in Section 2. Coherence resonance of HR neuron with GWN in situations before, near and after the Hopf bifurcation is presented in Section 3. In Section 4, noise-induced CS of two uncoupled identical HR neurons is studied, with a single neuron exhibiting quiescent state, periodic bursting, chaotic bursting or periodic spiking. Noise-enhanced CS of two weakly coupled identical neurons is discussed in Section 5. Finally, a conclusion is given in Section 6.

2. Deterministic and stochastic Hindmarsh–Rose neuron models

2.1. Deterministic HR neuron model

The three-variable HR model of action potential[18] was proposed as a mathematical representation of the firing behaviour of neurons, and it was originally introduced to give a bursting type with long interspike intervals (ISIs) of real neurons. It can be used to simulate spiking/bursting and chaos phenomena in real neurons. The equations of HR model are given as follows:

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= r(s(x - \chi) - z), \end{aligned} \tag{1}$$

where x represents membrane action potential, y is a recovery variable associated with the fast current, Na^+ or K^+, and z is a slow adaptation current such as Ca^{2+}; and a, b, c, d, r, s and χ are constants; I is the external direct current. In the following numerical simulations, let $a = 1.0, b = 3.0, c = 1.0, d = 5.0, r = 0.006, s = 4.0, \chi = -1.6$ and treat I as a control parameter to produce various firing patterns.

The firing patterns of a single HR neuron under different external currents I was explored in Ref.[19] through bifurcation analysis of the ISIs and the fast/slow dynamical analysis. It was concluded that with increasing external current, a single HR neuron may experience different dynamical states such as quiescent state, periodic bursting, chaotic bursting, and periodic spiking. A bifurcation diagram of ISIs versus the current I is shown in Fig.1.

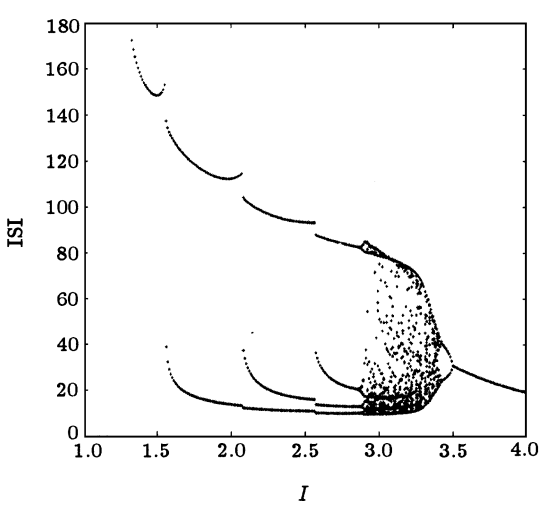


Fig.1. Bifurcation diagram of ISIs vs the additional direct current I in the single HR neuron model (cited from Ref.[19]).

2.2. Stochastic HR neuron model

The stochastic HR neuron model, in which a GWN is added to the membrane potential of the corresponding deterministic HR model, is given as follows:

$$\begin{aligned}\dot{x} &= y - ax^3 + bx^2 - z + I + \xi(t),\\ \dot{y} &= c - dx^2 - y,\\ \dot{z} &= r(s(x-\chi) - z),\end{aligned} \tag{2}$$

where $\xi(t)$ is GWN with the statistical property $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t')$, where D is the noise intensity and $\delta(\cdot)$ is the Dirac delta-function.

Longtin[7] observed autonomous stochastic resonance (ASR) in the subthreshold oscillation in the HR model near a bifurcation point when OUN was introduced to the model. Noise-induced bursting was observed and the interspike interval histogram (ISIH) exhibited multimodal mode, which meant integer multiple bursting. The effects of additive GWN, OUN and QMN on HR neuron model have been numerically studied and compared.[10] According to the dynamical behaviour of HR model near the critical Hopf bifurcation point, the origins of noise-induced burst and CR were analyzed, and it was concluded that with a proper choice of the noise intensity, CR can be exhibited in HR model. In the following section, we will consider CR of HR neurons subjected to a GWN under various situations, including periodic firing pattern after the Hopf bifurcation, as well as the quiescent state and subthreshold oscillation before the Hopf bifurcation.

3. Coherence resonance of a single HR neuron due to Gaussian white noise

According to Ref.[20], $I = 1.32$ is the Hopf bifurcation point of HR model for $r = 0.006$. Firstly, we study the CR of a single HR neuron due to GWN under situations before, near and after the Hopf bifurcation by choosing $I = 1.0$, $I = 1.31$ and $I = 1.4$, respectively, and investigate the corresponding firing patterns. For the cases without noise, the noiseless neuron exhibits the quiescent state, subthreshold oscillation and period-1 bursting when $I = 1.0, I = 1.31$ and 1.4, respectively. Then the noise can induce spiking sequences and the average ISI $\langle \mathrm{ISI}_i\rangle$ decreases as the noise intensity increases (see Fig.2(a)). This means that more intense noise induces spikes more frequently. For every given noise intensity, statistics were carried out by using 3×10^8 times of calculation with the time step 10^{-3}. A spike is counted if the membrane potential reaches 1.0. We also employ a measure of coherence proposed in Refs.[11,12], that is, the coefficient of variation defined as the ratio of the standard deviation of the ISIs to its mean: that is,

$$R = \frac{\sqrt{\mathrm{Var}(\mathrm{ISI}_i)}}{\langle \mathrm{ISI}_i\rangle}.$$

If R is close to zero, one may speak of a coherent output. The coherence resonance therefore manifests by a minimum of the coefficient of variation versus the noise intensity. It is obvious from Fig.2(b) that R first decreases, reaches a minimum value around $D = 0.45$ for $I = 1.0$, $D = 0.2$ for $I = 1.31$ and $D = 0.2$ for $I = 1.4$, and then increases for larger D. In Fig.3, we plot the ISI histogram (ISIH) and the return map of ISI with $I = 1.25, D = 0.05$, $I = 1.31, D = 0.03$ and

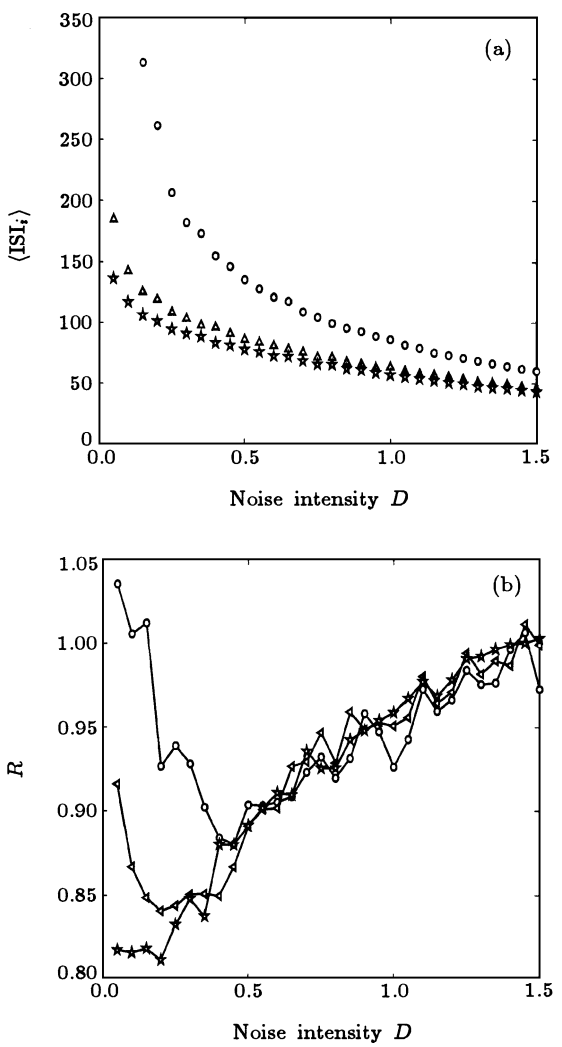


Fig.2. Diagrams of average ISIs $\langle \mathrm{ISI}_i\rangle$ (a) and coefficient of variation R (b) vs the noise intensity, when single neuron exhibits the quiescent state at $I = 1.0$ ($\circ$), subthreshold oscillation at $I = 1.31$ ($\triangleleft$) and period-1 bursting at $I = 1.4$ ($\star$).

$I = 1.35, D = 0.05$. All the ISIHs behave in multimodal mode, and the amplitude of peaks decay with increasing ISI in Figs.3(a)–(c). All the return maps of ISI have lattice-like structures in Figs.3(d)–(f). Since this behaviour is the manifestation of integer multiple spiking, hence, noise can induce integer multiple spiking not only near but also before and after the Hopf bifurcation.

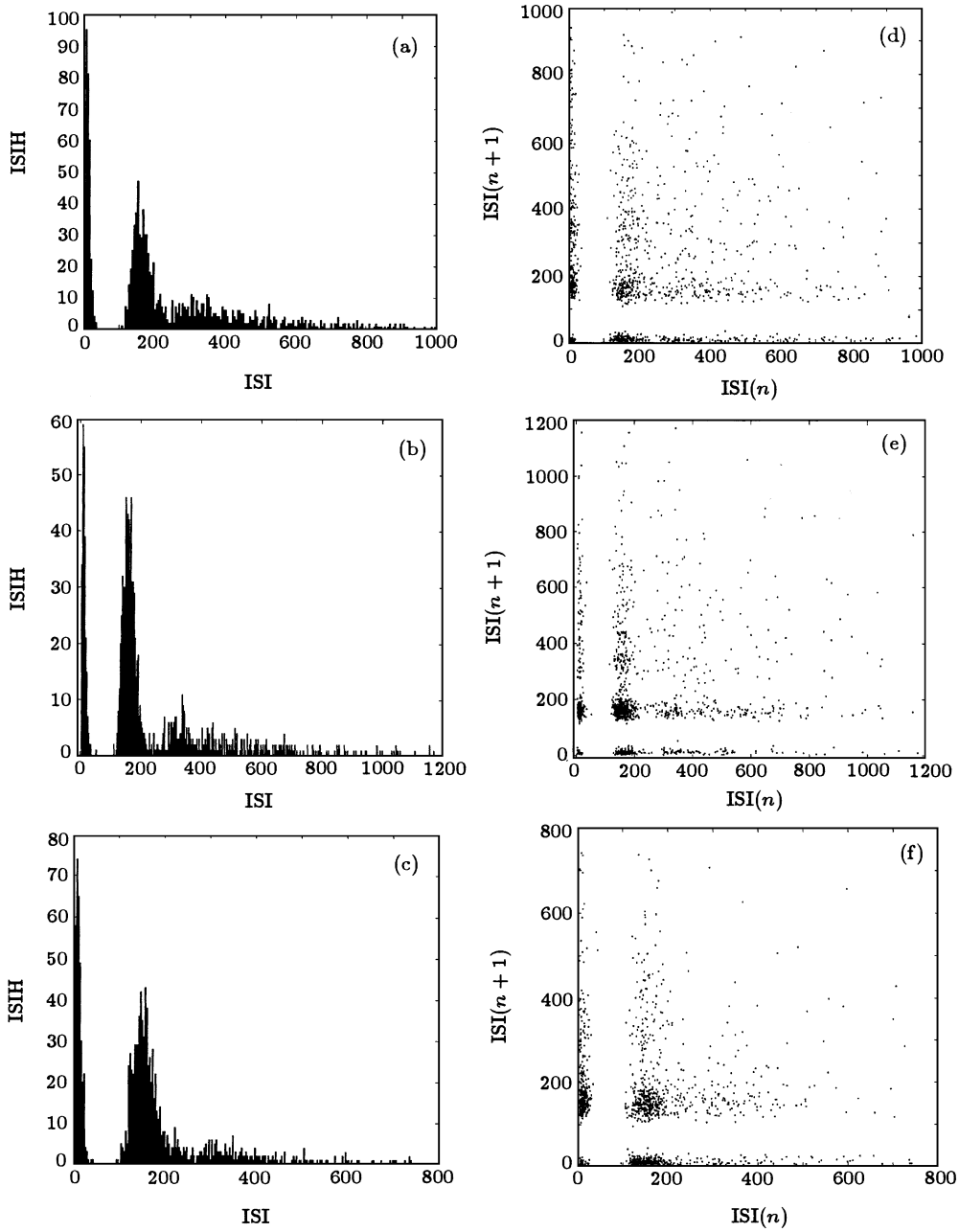


Fig.3. ISIH and return map of ISI for the HR neuron model with $I = 1.25, D = 0.05$ (a)(d), $I = 1.31, D = 0.03$ (b)(e) and $I = 1.35, D = 0.05$ (c)(f).

4. Noise-induced complete synchronization of uncoupled HR neurons

In neural systems, different neurons connected to another group of neurons may receive a common input signal, which often approaches a Gaussian distribution as a result of integration of many independent synaptic currents. Common synaptic current induces CS in neurons, which is of great importance for the information processing of neurons. A single neuron may faithfully encode temporal information in the timing

of successive spikes, and a group of neurons can respond collectively to a common synaptic current due to synchronization.

In general, when two identical neurons start from different initial conditions, the spike timings of the two neurons may not coincide. However, when a randomly fluctuating injection current, which simulates realistic, summed synaptic current, is applied to the two neurons, the spike timings can be driven into synchronization.[17]

It is important to emphasize that, in neuroscience, both the coupling and the random forcing are often rather weak among the elements. The basic geometrical structure of the trajectories of a single neuron in the phase space is preserved with weak noise perturbations, while it is destroyed significantly by strong noise. Hence, in what follows, we only consider the noise effects on CS with weak noise.

In this section, we study the synchronization of two uncoupled identical HR neurons driven randomly by a common additive GWN, i.e.

$$\begin{aligned} \dot{\boldsymbol{w}}_1 &= f(\boldsymbol{w}_1) + \boldsymbol{\xi}(t), \\ \dot{\boldsymbol{w}}_2 &= f(\boldsymbol{w}_2) + \boldsymbol{\xi}(t), \end{aligned} \tag{3}$$

where $\boldsymbol{w}_i = (x_i, y_i, z_i)(i = 1, 2)$ denotes the state variables of the i-th neuron and $\boldsymbol{\xi}(t) = (\xi(t), 0, 0)$, in which $\xi(t)$ is GWN as in Section 3. Here $f : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is a smooth vector function given by the right-hand side of Eq.(1), which is independent of i since the neurons are assumed to be identical.

Consider the synchronization error $\boldsymbol{e} = \boldsymbol{w}_2 - \boldsymbol{w}_1$, which is assumed to be small, and linearize the synchronization error system, then we obtain

$$\dot{\boldsymbol{e}} = Df(\boldsymbol{w})\boldsymbol{e}, \tag{4}$$

where $Df(\boldsymbol{w})$ stands for the Jacobian matrix on the trajectory of a neuron. This linearized equation has the same form as the noise-free system (1), while the trajectory of $\boldsymbol{w}$ here is different from that of the noise-free system. Based on the linearized dynamics, Lyapunov exponents are well defined[21] similar to that in the deterministic case. According to Eq.(4), two neurons will reach CS when the largest Lyapunov exponent becomes negative. We calculate the largest Lyapunov exponent λ_1 as a function of the noise intensity D for $I = 1.0, 1.31, 1.4, 3.0, 4.0$, which correspond respectively to the quiescent state, subthreshold oscillation, periodic bursting, chaotic bursting and periodic spiking of neurons without noise. In parallel, the mean synchronization error between two neurons defined by $\langle \boldsymbol{e} \rangle = \langle \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \rangle$, in which the angle bracket means time averaging, is also computed. It is seen from Fig.4(a) that λ_1 undergoes a transition from positive to negative at a critical noise intensity. Beyond this critical point, two identical HR neurons with the same input converge to an identical spike sequence after a short transient time, and the synchronization error vanishes as shown in Fig.4(b).

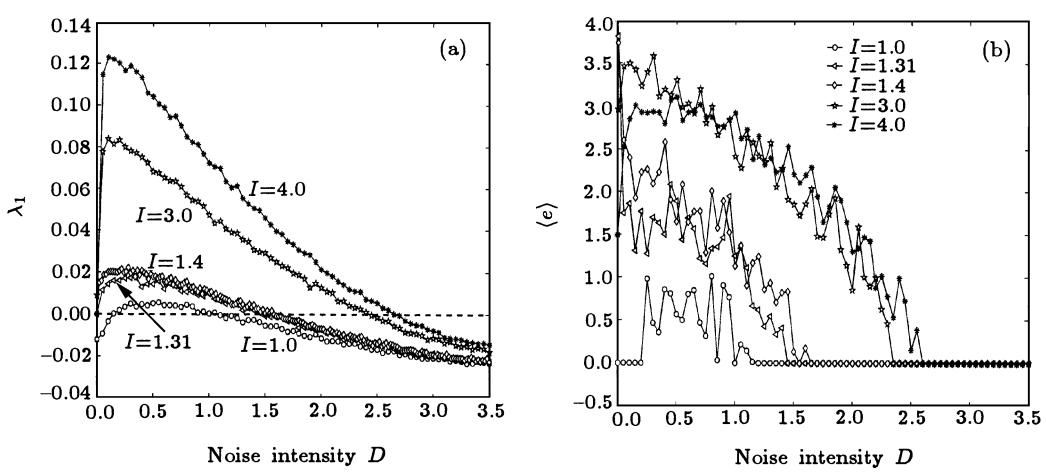


Fig.4. (a) The largest Lyapunov exponents λ_1 vs the noise intensity; (b) the mean synchronization error $\langle \boldsymbol{e} \rangle$ vs the noise intensity for $I = 1.0$ ($\circ$), $I = 1.31$ ($\triangleleft$), $I = 1.4$ ($\diamond$), $I = 3.0$ ($\star$) and $I = 4.0$ ($*$).

The key mechanism of noise-induced CS is that the existence of a saddle-focus or a saddle embedded in the periodic orbit or the chaotic orbit in the phase space. Noise changes the competition between contraction and expansion of the stable and the unstable manifolds for the saddle, and makes the contraction

dominant, so that the systems with different initial conditions converge to an identical response. When the density of the noise is large enough such that all the eigenvalues of the Jacobian matrix $Df(\boldsymbol{w})$ have a negative real part, that is, all the Lyapunov exponents are smaller than zero, which means that the nearby trajectories converge to each other, CS would occur.

5. Noise-enhanced complete synchronization of coupled HR neurons

In fact, as in neuroscience, systems are often coupled besides the common random forcing. In this section, we study the interplay between noise and the coupling of the systems to the effect of synchronization. In Ref.[19], we have considered CS of two coupled HR neurons without noise and derived the critical values of coupling strength for different firing patterns of neuron with varying external current. For weak coupling strength such as $C = 0.1$, two coupled identical HR neurons will not reach CS when they are out of a quiescent state. Under this situation, we consider the effect of the common noise added to them.

Considering two electrically coupled HR neurons driven by a common GWN, the network model reads

$$\begin{aligned} \dot{\boldsymbol{w}}_1 &= f(\boldsymbol{w}_1) + C(x_2 - x_1) + \boldsymbol{\xi}(t), \\ \dot{\boldsymbol{w}}_2 &= f(\boldsymbol{w}_2) + C(x_1 - x_2) + \boldsymbol{\xi}(t), \end{aligned} \tag{5}$$

where the vector function f is given by the right-hand side of Eq.(1) and $\boldsymbol{\xi}(t)$ is the same as in Section 4.

Similarly, for synchronization error $\boldsymbol{e} = \boldsymbol{w}_2 - \boldsymbol{w}_1$, which is assumed to be small, the linearized equations for Eq.(5) are

$$\dot{\boldsymbol{e}} = (Df(\boldsymbol{w}) - H)\boldsymbol{e}, H = \begin{pmatrix} 2C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

which is the same as that of noiseless situation, while the trajectory of $\boldsymbol{w}$ is determined by Eq.(5) and is noise-correlated. In terms of Eq.(6), two neurons will reach CS when the largest conditional Lyapunov exponent becomes negative. The largest conditional Lyapunov exponent λ_1^c and the mean synchronization error $\langle \boldsymbol{e} \rangle$ is calculated as functions of the noise intensity D. From Fig.5(a) it can be seen that λ_1^c undergoes a transition from positive to negative at a critical noise intensity, and the synchronization error vanishes beyond this critical value, as shown in Fig.5(b).

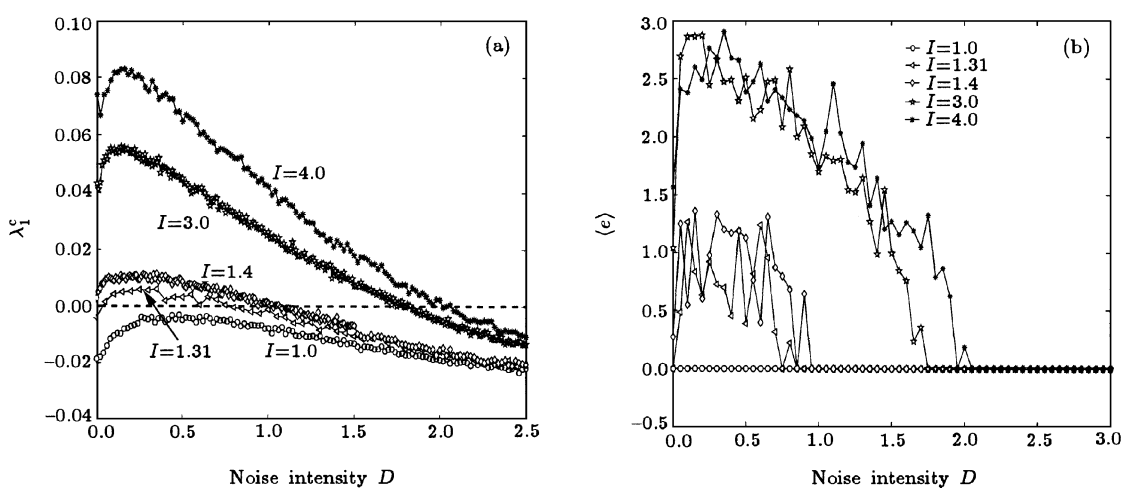


Fig.5. (a) The largest conditional Lyapunov exponents λ_1^c vs the noise intensity; (b) the mean synchronization error $\langle e \rangle$ vs the noise intensity when $I = 1.0$ ($\circ$), $I = 1.31$ ($\triangleleft$), $I = 1.4$ ($\diamond$), $I = 3.0$ ($\star$) and $I = 4.0$ ($*$).

Finally, we compute the critical values of the noise intensity with respect to the external direct current, which correspond to various firing patterns of a single neuron under uncoupled and coupled situations. It is obvious from Fig.6 that the critical value of the noise intensity increases with the external direct current for two uncoupled or weakly coupled HR neurons. Recalling the variation of the firing activities with the external direct current described in Section 2: that is, with increasing current I, the firing pattern of a single HR neuron transits from bursting to spiking; this result means that the bursting firing pattern is easier to

induce synchronization by noise than the spiking firing pattern. With coupling, the critical values of noise intensity are less than those without coupling, which implies that weak coupling enhances the sensitivity of synchronization of two neurons to the noise stimulus.

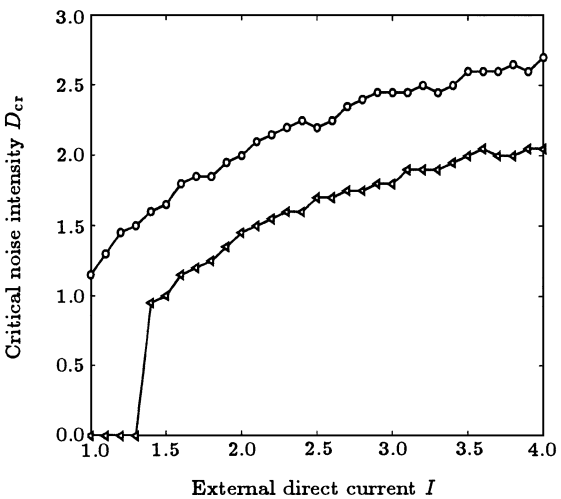


Fig.6. The critical values of the noise intensity for perfect synchronization of two uncoupled HR neurons (∘) and two weakly coupled HR neurons (◁) with a common GWN vs the external direct current added to each neuron.

6. Conclusion

In this article, we have studied the effects of noise on CR and synchronization of HR neurons. The CR of a single HR neuron under situations before, near and after the Hopf bifurcation was discussed when the external direct current I was treated as a bifurcation parameter. Integer multiple spiking caused by weak noise was found not only near but also before and after the Hopf bifurcation. We employed the coefficient of variation R to evaluate CR and found that it first decreased, reached a minimum and then increased with the increase of the noise intensity.

It is also demonstrated that noise is able to induce and enhance CS for two uncoupled or weakly coupled identical neurons. For noise induced CS in uncoupled neurons, the largest Lyapunov exponent λ_1 firstly increases, reaches a maximum value, and then decreases with the increase of the noise intensity; it finally undergoes a transition from positive to negative at a critical noise intensity, which means reaching CS. For two weakly coupled neurons driven by a common noise, the largest conditional Lyapunov exponent λ_1^c experiences the same transition, and it becomes negative from positive at a critical noise intensity for reaching CS.

Furthermore, the critical values of noise intensity were calculated for uncoupled and weakly coupled identical HR neurons for CS. It is concluded that the critical noise intensity increases with the increase of the external direct current. This implies that it is easier for noise to induce synchronization in bursting neurons than in spiking ones. It is also shown that the critical noise intensity for weakly coupled neurons are less than those for uncoupled neurons. This means that weak coupling can enhance the sensitivity of synchronization of neurons to the noise stimulus.

Because the firing activities of bursting and synchronization of neurons are very important in the processes of information generation and transfer in neural systems, the results of CR and synchronization of HR neurons with noise in this article may facilitate to understand the effects of noise in information processing of neural systems.

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