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Stochastic resonance in an asymmetric bistable system driven by multiplicative and additive noise*

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The phenomenon of stochastic resonance (SR) in a bistable system driven by multiplicative and additive white noises and a periodic rectangular signal with a constant component is studied according to the theory of signal-to-noise ratio (SNR) in the adiabatic limit. The analytic expression of the SNR is obtained for arbitrary signal amplitude without being restricted to small amplitudes. We find that the effects of the multiplicative noise intensity D and additive noise intensity α on SNR are different: the SNR-α curve shows SR for almost the whole range of static asymmetry r, but the SNR-D curve only displays SR for some values of static asymmetry r. It is more sensitive to control SR through adjusting the additive noise intensity α than adjusting the multiplicative noise intensity D, when the asymmetry of bistable potential r is not too large. Moreover, the static asymmetry r decreases the SNR.

Keywords: stochastic resonance, periodic rectangular signal, signal-to-noise ratio
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1. Introduction

Benzi *et al*[1] and Nicolis and Nicolis[2] first proposed the term "stochastic resonance" (SR) to explain the periodic recurrences of the earth's ice ages. There are many theoretical and experimental papers appearing thereafter. Fauve and Heslot[3] and McNamara *et al*[4] observed the SR through experiments. Meanwhile, SR has been investigated theoretically.[5−10] McNamara *et al*[4,5] suggested a two-state model and obtained the signal-to-noise ratio (SNR) in adiabatic limit to characterize the SR. Dykman *et al*[6] and Hu *et al*[7] introduced the linear-response theory and perturbation theory to investigate the SR. Zhou *et al*[8] employed the residence-time distribution to explain the SR as a resonance synchronization phenomenon. Based on these theories, researchers considered the correlation between noises and explored the effects of noises on SNR in the bistable system with the periodic modulated signal.[11−18] However, the expression of SNR obtained under the adiabatic condition is only valid for small signal amplitude. Later, by selecting a proper periodic rectangular signal, one can solve the above problem for any value of signal amplitude, without being restricted to small signal amplitudes.[19,20] Gerashchenko[20] has investigated the phenomenon of SR in an asymmetric bistable system driven by additive white noise and a periodic rectangular signal theoretically and experimentally. It is well known that the multiplicative noise often plays a different role on SNR from the additive noise. Therefore, the multiplicative noise is introduced to the bistable system driven by additive white noise and a periodic rectangular signal in our paper.

In this paper, we study the SR in a bistable system driven by multiplicative and additive white noises and a periodic rectangular signal with a constant component. The increase of the constant component leads to a static asymmetry of the bistable potential. An analytical expression of SNR is obtained in the adiabatic limit for arbitrary signal amplitudes. According to the expression of SNR, we find that the effects of multiplicative noise intensity D and additive noise in-

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tensity α on SNR are different from each other. The SNR-α curve shows SR over almost the whole range of static asymmetry r, the peak height becomes lower and the position of the peak moves to the right with increasing r. But the SNR-D curve only displays SR for some values of static asymmetry r. The curve of SNR versus the noise intensity ratio $R = D/\alpha$ exhibits a pronounced single-peak and SR appears. The peak height decreases with increasing r and the position of peak moves to the left. Therefore, the static asymmetry r decreases the SNR.

2. The SNR for a bistable system

Consider a one-dimensional bistable stochastic system driven by multiplicative white noise $\xi(t)$, additive white noise $\eta(t)$ and a periodic rectangular signal $AG(t)$. The model is generally governed by the following Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial U_0(x)}{\partial x} + r + x\xi(t) + \eta(t) + AG(t), \quad (1)$$

$$U_0(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4, \quad (2)$$

where $U_0(x)$ denotes a symmetric bistable potential function, and the constant r characterizes the bias of the bistable system. A is a constant. The periodic rectangular signal $G(t)$ with period T is given by

$$G(t+T) = G(t) = \begin{cases} 1, & 0 < t \le T/2, \\ -1, & T/2 < t \le T. \end{cases} \quad (3)$$

The noise terms $\xi(t)$ and $\eta(t)$ are uncorrelated white noises, and their statistical properties are characterized by their mean and variance

$$\begin{aligned} &\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \\ &\langle \xi(t)\xi(t') \rangle = 2D\delta(t-t'), \\ &\langle \eta(t)\eta(t') \rangle = 2\alpha\delta(t-t') \\ &\langle \xi(t)\eta(t') \rangle = \langle \eta(t)\xi(t') \rangle = 0, \end{aligned} \quad (4)$$

where α and D are the additive and multiplicative noise intensities respectively.

According to Eqs.(1)–(4), the stationary probability distribution (SPD) can be written as

$$\rho_{st}(x) = N[B(x)]^{-1/2} \exp\left[-\frac{\tilde{U}(x)}{D}\right], \quad (5)$$

where N is the normalization constant, $B(x) = Dx^2 + \alpha$, and the modified potential is

$$\tilde{U}(x) = \int_{-\infty}^{x} \mathrm{d}x_1 \{[-U_0'(x_1) + r + AG(t)]D/B(x_1)\}.$$

Assume that $x_\pm = \pm\sqrt{a/b}$ and $x_0 = 0$ are the stable and unstable points in the unperturbed system. Under the adiabatic condition and the time scale $T \gg w_0^{-1}$, the transition rate out of the $x_\pm$ can be obtained:

$$\begin{aligned} W_\pm =& \frac{|U_0''(x_0)U_0''(x_\pm)|^{1/2}}{2\pi} \exp\left[\frac{\tilde{U}(x_\pm) - \tilde{U}(x_0)}{D}\right] \\ =& w_0 \exp(\mp p \mp qG(t)), \end{aligned} \quad (6)$$

where w_0 stands for the characteristic frequency of switching when Eq. (1) is only driven by additive and multiplicative white noises, and the expressions of w_0, p and q are given by

$$\begin{aligned} w_0 =& \frac{a}{\sqrt{2}\pi} \exp\left[-\frac{(a + b/R)\ln(aR/b+1) - a}{2D}\right], \\ p =& \frac{r\sqrt{R}\arctan(\sqrt{aR/b})}{D}, \\ q =& \frac{A\sqrt{R}\arctan(\sqrt{aR/b})}{D}, \quad R = \frac{D}{\alpha}. \end{aligned} \quad (7)$$

From Ref.[5], the bistable case can be regarded as a two-state system, with $n_\pm$ denoting the occupation probabilities of the stable states $x_\pm$. The master equation for these occupation probabilities $n_\pm$ is found to be

$$\dot{n}_+ = -\dot{n}_- = W_-(t)n_- - W_+(t)n_+. \quad (8)$$

Under the adiabatic conditions, the local balance in the bistable model is established much faster than the transition probabilities change. From Eq.(8), one can obtain the following results:

$$\begin{aligned} n_+(t) &= \frac{W_-(t)}{W_-(t) + W_+(t)}, \\ n_-(t) &= \frac{W_+(t)}{W_-(t) + W_+(t)}. \end{aligned} \quad (9)$$

The following formula is valid for the arbitrary function f:

$$\begin{aligned} f[r + AG(t)] =& \frac{1}{2}[f(r+A) + f(r-A) \\ &+ G(t)(f(r+A) - f(r-A))]. \end{aligned} \quad (10)$$

By using formula (10) and combining with the method in Refs.[19,20], the expression of the correlation function has the following form:

$$K(t) = B_2(p,q)\langle G(t)G(0) \rangle + C(p,q)\delta(t), \quad (11)$$

where

$$\langle G(t)G(0)\rangle = \frac{4}{\pi^2}\sum_{k=0}^{\infty}(2k+1)^{-2}\exp[-\mathrm{i}(2k+1)\Omega t],$$

$$\Omega = \frac{2\pi}{T},$$

$$B_2(p,q) = \frac{a}{4b}(\tanh(p+q) - \tanh(p-q))^2,$$

$$C(p,q) = \frac{a}{2bw_0}\left(\frac{1}{\cosh^3(p+q)} + \frac{1}{\cosh^3(p-q)}\right).$$

The power spectrum of fluctuations $S(\omega)$, defined as the Fourier transform of the correlation function, can be given by

$$S(\omega) = S_1(0) + S_2(\omega), \tag{12}$$

where

$$\begin{aligned} S_1(0) =& C(p,q), \\ S_2(\omega) =& B_2(p,q)\frac{8}{\pi} \\ & \times \sum_{k=0}^{\infty}(2k+1)^{-2}\delta[\omega-(2k+1)\Omega]. \end{aligned}$$

Here $S_1(0)$ is the power density at the zero frequency associated with the noise background in the adiabatic limit case. $S_2(\omega)$ is the power density connected with the output signal. Then the expression of SNR, defined as the ratio of the power density of signal to the noise background, is

$$\mathrm{SNR} = \frac{8}{\pi}\cdot\frac{B_2(p,q)}{C(p,q)}. \tag{13}$$

It should be pointed out that formula (13) is derived in the adiabatic limit, and the condition $q \ll 1$ is satisfied. Hence the expression of SNR can be simplified as

$$\mathrm{SNR} = \frac{8}{\pi}\cdot\frac{w_0 q^2}{\cosh(p)}, \tag{14}$$

where the functions p and q have been defined in Eq.(7).

3. Discussion and conclusion

Using Eq.(14), we discuss the influences of the additive noise intensity α, multiplicative noise intensity D, noise intensity ratio R and static asymmetry r on the SNR. It should be mentioned that the curves in Figs.1–5 are plotted in the case of $r > 0$ for simplicity.

In Fig.1, we present the SNR as a function of the additive noise intensity α for different values of r. The single peak on the SNR-α curve implies SR. With the increase of the static asymmetry r, the peak becomes lower and the position of the peak shifts to the right. Then how to understand the phenomena shown in Fig.1? We can explain it from the extremal curves of the SNR shown in Fig.2, the extremal points of the SNR only lie on the extremal curve corresponding to $D = 4$. Obviously, with the increase of r, α is also increased correspondingly. This is why the position of resonance peak moves to the right with the increase of r in Fig.1.

In Fig.1 the value of multiplicative noise intensity D is fixed. Now an interesting point is the behaviour of the extremal curve with the variation of multiplicative noise intensity D. The extremal curves for different values of D are shown in Fig.2. The extremal curve moves upward when D is increased. Meanwhile, the corresponding SNR-α curves have some small changes: the position of the peak on the SNR-α curves moves to the left and the peak becomes a little higher for fixed asymmetry r with the increase of D. For brevity, we do not present the SNR-α curves correspond to $D = 2$ and 6 here.

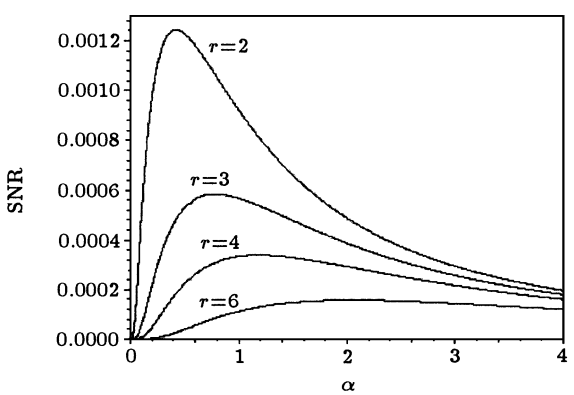


Fig.1. SNR as a function of the additive noise intensity α for $a = b = 1$, $A = 0.1$ and $D = 4$ for different values of r.

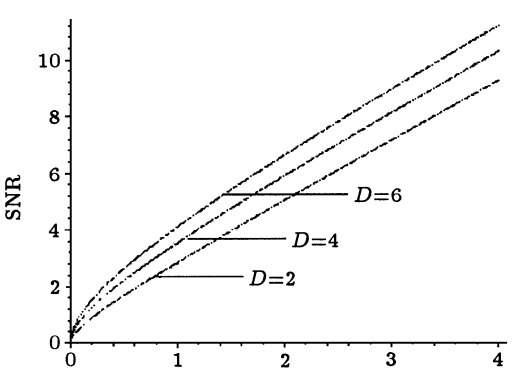


Fig.2. The extremal curve of the SNR with parameters $a = b = 1$, $A = 0.1$ and $D = 2$, 4, 6.

We plot the curves of the SNR versus D with different values of r in Fig.3. In Fig.3(a), by fixing the values of a, b, A and α, SNR decreases monotonically with increasing D when the static asymmetry $r < 4$. While the static asymmetry $r \geq 4$, there is a small peak on the SNR-D curve, and SNR is a nonmonotonic function of D. SR occurs in this case, as shown in Fig.3(b). Figure 4 can help us to understand the phenomena shown in Fig.3. It is impossible for the SNR to attain the extremum when the point (D, r) is out of the extremal curve plotted in Fig.4. When we fix the additive noise intensity $\alpha = 2$, the SNR cannot attain the maximum in the case of $r < 4$. The extremal point (D, r) of the SNR is on the curve only in the case of $r \geq 4$.

The value of additive noise intensity α is fixed in Fig.3. We want to find whether the change of α will lead to the change of asymmetry r. In Fig.4, we plot the extremal curve when α is varied. For $\alpha = 1$ the SNR can reach the maximum only in the case of $r \geq 2$. When we fix $\alpha = 4$, the SR appears in the case of $r \geq 8$. So the value of asymmetry r for SNR being the maximum is increased when the additive noise intensity is increased.

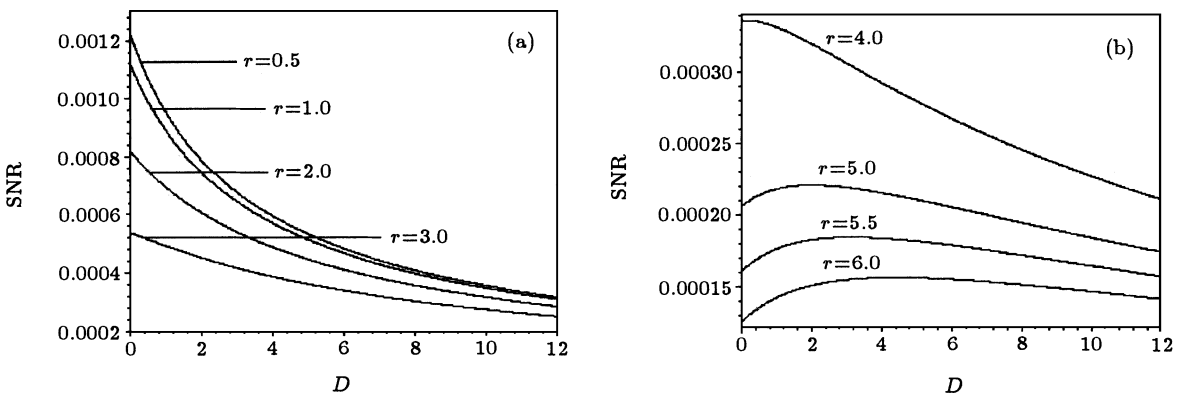


Fig.3. SNR as a function of the multiplicative noise intensity D for $a = b = 1$, $A = 0.1$ and $\alpha = 2$ are fixed. (a) $r = 0.5,\ 1,\ 2,\ 3$, (b) $r = 4,\ 5,\ 5.5,\ 6$.

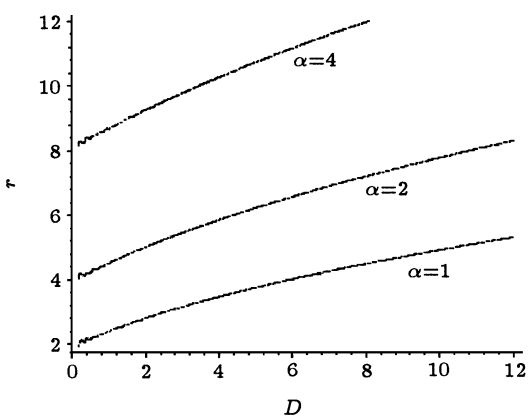


Fig.4. The extremal curve of SNR with parameters $a = b = 1$, $A = 0.1$ and $\alpha = 1,\ 2,\ 4$.

In Figs.1 and 3, one can see that the effects of the additive and multiplicative noises on the SNR are different. When the asymmetry r of the bistable potential is not too large, the change of additive noise intensity α is easier to produce SR than the multiplicative noise intensity D.

In Fig.5 the SNR is shown as a function of the noise intensity ratio $R = D/\alpha$ for different values of r. The curve exhibits a pronounced single-peak and SR appears on the R-SNR parameter plane. When asymmetry r is increased, the peak becomes lower and the position of the peak moves to the left. The curve of SNR versus the static asymmetry r is given in Fig.6. Obviously, the SNR is decreased when $|r|$ is increased. The increase of the static asymmetry $|r|$ decreases the SNR, which is consistent with the result of Ref.[20]. Meanwhile, we can also see that the SNR is increased when the noise intensity ratio R is increased as shown in Fig.6.

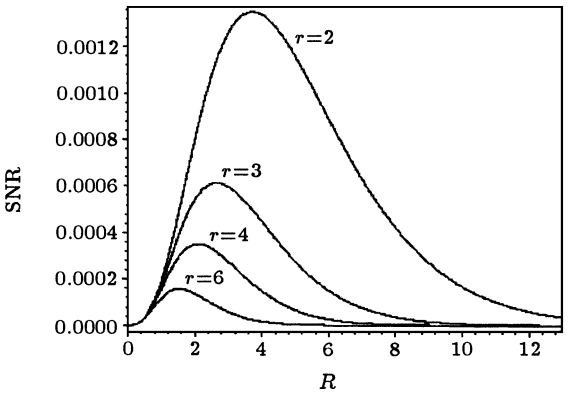


Fig.5. SNR as a function of the noise intensity ratio $R = D/\alpha$ for $a = b = 1$, $A = 0.1$ and $D = 4$ for different values of r.

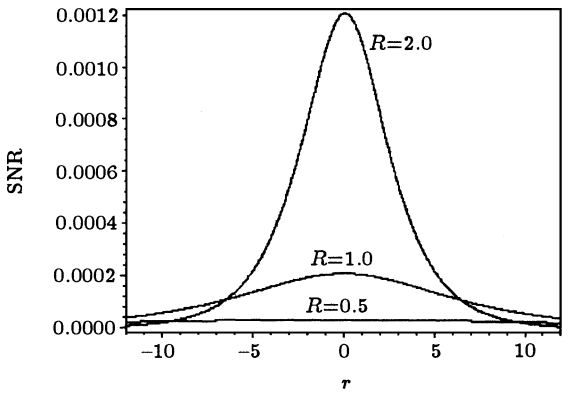


Fig.6. SNR as a function of the static asymmetry r for $a = b = 1$, $A = 0.1$ and $D = 4$ for R=0.5, 1, 2.

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