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Exact solutions of N-dimensional harmonic oscillator via Laplace transformation

Chen Gang(陈 刚)†

Department of physics, Shaoxing College of Arts and Sciences, Shaoxing 312000, *China*

Materials Physics and Equipments Institute of Shaoxing University, Shaoxing 312000, *China*

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The N-dimensional Schrödinger equation for the harmonic oscillator is reduced to a first-order differential equation in terms of the Laplace transformation and the exact bound state solutions are derived. It is shown that this method of solving Schrödinger equation may serve as a substitute for the standard functional, analytical approach also in lower dimensions.

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It goes without saying that the exact bound state solutions of the Schrödinger equation with physical potentials play an important role in quantum mechanics. Discussions about the method to solve the Schrödinger equation have aroused much interest. All kinds of methods, such as the factorization,[1] the path integral,[2] the algebraic method,[3] the $1/N$ expansion,[4] the analytic continuation,[5] the eigenvalue moment method,[6] the power series expansion[7] and the supersymmetric quantum mechanics[8] methods, can be enumerated amongst others of the solution of the wave equations. Laplace transformation, which is integral transformation, is comprehensively useful in physics and engineering.[9] It reduces a second-order differential equation to a first-order differential equation. In quantum mechanics, such techniques were used by Schrödinger in his first paper on the quantum mechanical hydrogen atom[10] and then by Swainson and Drake on the recurrence relations of radial wavefunctions for the hydrogen atom.[11] Recently, the exact bound state solutions of the Schrödinger equation with the Morse potential were also obtained by the author.[12]

In modern theory, higher spatial dimensions are of interest. Recently, the N-dimensional Schrödinger equations with central potentials have been discussed by many authors.[13−16] In this article we employ the Laplace transformation to solve the Schrödinger equation of the N-dimensional harmonic oscillator and the exact bound state solutions are obtained.

For a particle in an N-dimensional Euclidian space, the time independent radial Schrödinger equation is shown as[13−16]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{L(L+N-2)}{r^2}-\frac{2m}{\hbar^2}V(r)+\frac{2m}{\hbar^2}E\right]R(r)=0. \tag{1}$$

For an isotropic harmonic oscillator of angular frequency ω the potential function is given by

$$V(r)=\frac{1}{2}m\omega^2 r^2. \tag{2}$$

Substitution of Eq.(2) into Eq.(1) yields

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{L(L+N-2)}{r^2}-\frac{m^2\omega^2}{\hbar^2}r^2+\frac{2m}{\hbar^2}E\right]R(r)=0. \tag{3}$$

If we put

$$y=r^2, \tag{4}$$

$$\alpha=\frac{m\omega}{2\hbar}, \tag{5}$$

$$\beta^2=\frac{mE}{2\hbar^2}, \tag{6}$$

Eq.(3) turns into

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{N}{2}\frac{\mathrm{d}}{\mathrm{d}y}-\frac{L(L+N-2)}{4y}-\alpha^2 y+\beta^2\right]R(y)=0. \tag{7}$$

†E-mail: Chengang@zscas.edu.cn

In order to eliminate the polynomial $L(L+N-2)/4y$, we assume

$$R(y) = y^{-\frac{(L+N-2)}{2}} f(y) \tag{8}$$

and then insert it into Eq.(7) to obtain

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \left(L+\frac{N}{2}-2\right)\frac{\mathrm{d}}{\mathrm{d}y} - \alpha^2 y + \beta^2\right] f(y) = 0. \tag{9}$$

The Laplace transformation

$$L[g(t)] = \int_0^\infty g(t)\mathrm{e}^{-pt}\mathrm{d}t = G(p)^{[9]}$$

leads to an equation

$$(p^2-\alpha^2)\frac{\mathrm{d}}{\mathrm{d}p}F(p) + \left[\left(L+\frac{N}{2}\right)p - \beta^2\right]F(p) = 0. \tag{10}$$

Equation (10) is a first-order differential equation and therefore we may directly make use of the integral to get the expression

$$F(p) = C''(p+\alpha)^{-(L+N/2)}\left(\frac{p-\alpha}{p+\alpha}\right)^{\frac{\beta^2-(L+N/2)\alpha}{2\alpha}}, \tag{11}$$

where $C''$is a constant. Noting that

$$\left(\frac{p-\alpha}{p+\alpha}\right)^{\frac{\beta^2-(L+N/2)\alpha}{2\alpha}}$$

is a multi-valued function and the wavefunctions are required to be single-valued, we must take

$$\frac{\beta^2-(L+N/2)\alpha}{2\alpha} = n, \quad n = 0,1,2,3,... \tag{12}$$

Applying a simple series expansion to Eq.(12), we obtain

$$F(p) = C'\sum_{j=0}^{n}\frac{(-2\alpha)^j n!}{(n-j)!\,j!}(p+\alpha)^{-(L+N/2+j)}, \tag{13}$$

where C' is a constant. In terms of a simple extension of the inverse Laplace transformation[9] we can immediately deduce that

$$f(y) = Cy^{L+N/2-1}\mathrm{e}^{-\alpha y} \times \sum_{j=0}^{n}\frac{(-1)^j n!\Gamma(L+N/2)}{(n-j)!j!\Gamma(L+N/2+j)}(2\alpha y)^j. \tag{14}$$

where C is a constant. Comparing Eq.(14) with the series expansion of confluent hypergeometric functions

$$F(-n,\gamma,z) = \sum_{j=0}^{n}\frac{(-1)^j n!\Gamma(\gamma)}{(n-j)!j!\Gamma(\gamma+j)}z^j, \tag{15}$$

we obtain

$$f(y) = Cy^{L/2}\mathrm{e}^{-\alpha y}F(-n, L+N/2, 2\alpha y). \tag{16}$$

Noting Eqs.(4) and (8), we finally get the following radial wavefunctions:

$$R_{n,L} = Cr^L\exp\left[-\frac{m\omega}{2\hbar}r^2\right]F\left(-n, L+N/2, \frac{m\omega}{\hbar}r^2\right). \tag{17}$$

The normalization condition of radial wavefunctions

$$\int_0^\infty R^*_{n,L}(r)R_{n,L}(r)r^{N-1}\mathrm{d}r = 1 \tag{18}$$

and the integral formula of confluent hypergeometric functions

$$\int_0^\infty \mathrm{e}^{-z}z^\lambda F(-n,\lambda+1,z)^2\mathrm{d}z = \frac{\Gamma(\lambda+1)^2 n!}{\Gamma(\lambda+n+1)} \tag{19}$$

lead to the following expression:

$$C = \left[\frac{2\Gamma(L+N/2+n)}{\Gamma(L+N/2)^2 n!}\left(\frac{m\omega}{\hbar}\right)^{2L+N-1}\right]^{1/2}. \tag{20}$$

In terms of Eqs.(6) and (12), the bound state energy spectrum is shown as

$$E_{n,L} = (2n+L+N/2)\hbar\omega. \tag{21}$$

In general, the exact solutions of the Schrödinger equation of the N-dimensional harmonic oscillator are obtained in detail via the Laplace transformation. It is shown that this procedure is easier than previous ones and serves as a substitute for the standard functional analytical approach.

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