Vol 14 No 6, June 2005
1009-1963/2005/14(06)/1069-06

Chinese Physics

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Exact discrete soliton solutions of quintic discrete nonlinear Schrödinger equation*

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(Received 29 October 2004; revised manuscript received 18 January 2005)

By using the extended hyperbolic function approach, we have studied a quintic discrete nonlinear Schrödinger equation and obtained new exact localized solutions, including the discrete bright soliton solution, dark soliton solution, alternating phase bright soliton solution and alternating phase dark soliton solution, if a special constraint is imposed on the coefficients of the equation.

Keywords: discrete solitons, extended hyperbolic function approach
PACC: 0340K, 0290, 0230

1. Introduction

Discrete solitons in nonlinear lattices have been the focus of considerable attention in diverse branches of science.[1] Discrete solitons are possible in several physics fields, such as biological systems,[2] atomic chains,[3] solid state physics,[4] electrical lattices[5] and Bose–Einstein condensates.[6] Recently, the existence of discrete solitons in photonic structures (in arrays of coupled nonlinear optical waveguides[7,8] and in the nonlinear photonic crystal structure[9]) was announced and has attracted considerable attention in the scientific community. It is possible to make discrete waveguides using photonic crystals. In this situation, "discrete solitons" appear naturally and have interesting properties. Many scientists believe that discrete solitons can play an important role in this technology.

Here, we consider the physical system described by the discrete nonlinear Schrödinger (DNLS) equation. Most relevant for realistic applications (particularly to nonlinear optics) is the DNLS equation with a cubic (Kerr) nonlinearity:

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n = 0, \quad (1)$$

where ψ_n are complex variables defined for all integer values of the site index n. The DNLS equation was used by Christodoulides and Joseph[10] to model the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides. There are discrete spatial solutions. Experimental results for optical waveguide arrays, confirming the validity of the model, have been reported by Eisenberg *et al.*[11] There are many works about the DNLS equation and some are its modifications. It is well known that the standard DNLS equation is not completely integrable.[12]

Recently, attention has been attracted to discrete solitons in dissipative lattices. An important example of this is the integrable discrete nonlinear Schrödinger equation (Ablowitz–Ladik (AL) system),

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0, \quad (2)$$

which is proposed by using inverse scattering method.[13] The AL system has N-soliton solutions and a rich mathematical structure. However, it is not physically realistic, because it dose not contain the Kerr-nonlinearity term. Thus the deformed DNLS equation

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n + F(\psi_{n+1}, \psi_n, \psi_{n-1}) = 0, \quad (3)$$

*Project supported by the National Natural Science Foundation of China (Grant No 10372094), the Natural Science Foundation of Zhejiang Province, China (Grant No 102053) and the Foundation of Zhejiang Education Committee (Grant No 20030706).
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where $F(\psi_{n+1}, \psi_n, \psi_{n-1})$ is a polynomial of ψ_{n+1}, ψ_n, ψ_{n-1}, was proposed in several works. For example,

$$F(\psi_{n+1}, \psi_n, \psi_{n-1}) \equiv \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \quad (4)$$

was considered by several authors.[14−17] One can see that, for $\gamma = 0$, Eq.(3) together with Eq.(4) reduces to the standard DNLS equation (1), while for $\beta \to 0$ it gives the AL system. Kivshar and Salerno explained the physical meaning of Eq.(3) with Eq.(4).[15] They proposed that an application of Eq.(3) with Eq.(4) may be found in nonlinear optics, where the DNLS equation describes the interactions of partial TE(transverse electric) modes in an array of (focusing or defocusing) waveguides.

The existence of localized solutions is well known for spatially discrete systems(including quintic and higher-order nonlinearities).[18−20] However, to our knowledge, no one has found exact soliton-like solutions of Eq.(3)-type discrete systems except Maruno *et al.*[21] Concrete forms of localized solutions are, definitely, useful for analysing real physics. Here, we consider a model of the propagation of discrete self-trapped beams in an array of weakly-coupled nonlinear optical waveguides, the quintic discrete nonlinear Schrödinger (QDNLS) equation

$$\begin{aligned} &\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \alpha(\psi_{n+1} - 2\psi_n + \psi_{n-1}) + \beta|\psi_n|^2\psi_n \\ &+ \gamma|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \\ &+ \delta|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = 0. \end{aligned} \quad (5)$$

In physics problems, the quintic nonlinearity can be equal to or even more important than the cubic one[22] as it is responsible for the stability of localized solutions. Maruno *et al* obtained two exact solutions of Eq.(5) by using standard Hirota method.[21]

In recent decades, many powerful methods have been proposed, such as, inverse scattering method,[23] Painlevé truncation expansion,[24] homogeneous balance method,[25] tanh method,[26,27] Jacobian elliptic function method,[28,29] mapping deformation method,[30−32] and so on. However, to our knowledge,only a few works have been done to investigate exact solutions of a nonlinear differential-difference equation(NDDE),[33] because to extend the above methods to a NDDE is rather difficult. More recently, Baldwin *et al*[34] presented an algorithm to find exact travelling wave solutions of NDDE in terms of tanh function and found kink-type solutions in many spatially discrete nonlinear models such as Ablowitz–Ladik lattice, Toda lattice, Volterra lattice, discrete mKdV lattice, Hybrid lattice. However, this method is unable to find solutions other than polynomials in tanh. Can we find bell-type or other types of excitations for NDDE? We will further extend this hyperbolic function approach. We not only have found the kink-shaped soliton solutions of NDDE, but also found the bell-shaped soliton solutions of NDDE. In this paper, we investigate the QDNLS equation using this extended hyperbolic function approach. We not only obtain bright and dark soliton solutions of QDNLS equation (5), but also find new soliton solutions: namely, alternating phase bright and dark solitons.

The rest of this paper is organized as follows. In Section 2, we describe this extended hyperbolic function approach in detail. In Section 3, applying this extended approach, with the help of Maple, study and analyse Eq.(5). The last section is a short summary and discussion.

2. The extended hyperbolic function approach for NDDE

Given a system of M polynomial NDDE,

$$\begin{aligned} &\Delta(u_{n+p_1}(x), \cdots, u_{n+p_k}(x), u'_{n+p_1}(x), \cdots, \\ &u'_{n+p_k}(x), u^{(r)}_{n+p_1}(x), \cdots, u^{(r)}_{n+p_k}(x)) = 0, \end{aligned} \quad (6)$$

where the dependent variable u has M components u_i, the continuous variable x has N components x_i, the discrete variable n has Q components n_j, the k shift vectors are $\boldsymbol{P}_i$, and $\boldsymbol{u}^{(r)}(x)$ denotes the collection of mixed derivative of order r.

Step 1 Transform the NDDE into NDDE in T. We introduce the hyperbolic tangent

$$T_n = \tanh(\xi_n), \quad (7)$$

where

$$\xi_n = \sum_{i=1}^{Q} d_i n_i + \sum_{j=1}^{N} c_j x_j + \zeta, \quad (8)$$

where the coefficients $c_1, c_2, \cdots, c_N, d_1, d_2, \cdots, d_Q$ and the phase ζ are constants to be determined later. Repeatedly applying the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}x_j} = \frac{\partial\xi_n}{\partial x_j}\frac{\mathrm{d}T_n}{\mathrm{d}\xi_n}\frac{\mathrm{d}}{\mathrm{d}T_n} = c_j(1 - T_n^2)\frac{\mathrm{d}}{\mathrm{d}T_n}, \quad (9)$$

transforms Eq.(6) into

$$\begin{aligned} &\Delta(U_{n+p_1}(T_n), \cdots, U_{n+p_k}(T_n), U'_{n+p_1}(T_n), \cdots, \\ &U'_{n+p_k}(T_n), U^{(r)}_{n+p_1}(T_n), \cdots, U^{(r)}_{n+p_k}(T_n)) = 0. \end{aligned} \quad (10)$$

It is important to note that for any $s(s = 1, \cdots, k)$, U_{n+p_s} is a function of T_n and not T_{n+p_s}. Using the identity

$$\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}, \tag{11}$$

we can write

$$T_{n+p_s} = \frac{T_n + \tanh(\varphi_s)}{1 + T_n \tanh(\varphi_s)}, \tag{12}$$

where

$$\varphi_s = \boldsymbol{P}_s.d = p_{s1}d_1 + p_{s2}d_2 + \cdots + p_{sQ}d_Q, \tag{13}$$

and p_{sj} is the jth component of shift vector $\boldsymbol{P}_s$.

Step 2 Determine the degree of the polynomial solutions. Seeking solutions of the form

$$U_n(T_n) = a_0 + \sum_{i=1}^{m} T_n^{i-1}(a_i T_n + b_i\sqrt{1-T_n^2}), \tag{14}$$

where a_0, a_i, b_i are constants to be determined and m is fixed by balancing the highest-order derivative term with the highest-order nonlinear term in Eq.(10):

$$\begin{aligned}
&U_{n+p_s}(T_n)\\
=&a_0 + \sum_{i=1}^{m} T_{n+p_s}^{i-1}(a_i T_{n+p_s} + b_i\sqrt{1-T_{n+p_s}^2})\\
=&a_0 + \sum_{i=1}^{m}\left[\frac{T_n+\tanh(\varphi_s)}{1+T_n\tanh(\varphi_s)}\right]^{i-1}\\
&\times\left[a_i\frac{T_n+\tanh(\varphi_s)}{1+T_n\tanh(\varphi_s)}\right.\\
&\left.+ b_i\sqrt{1-\left(\frac{T_n+\tanh(\varphi_s)}{1+T_n\tanh(\varphi_s)}\right)^2}\right],
\end{aligned} \tag{15}$$

with φ_s given in Eq.(13).

step 3 Solve the nonlinear parametrized system. Substituting Eq.(14) with Eq.(15) into Eq.(10), collecting all terms with the same order of $T_n^i, T_n^i\sqrt{1-T_n^2}$ and setting the coefficients of each order of them to zero, yield a set of algebraic equations (AEs) in a_i, b_i and c_j. From these relations a_0, a_i, b_i, c_j can be determined.

3. Exact solutions of QDNLS equation

In order to obtain some exact solutions of Eq.(5), firstly we make the transformation

$$\psi_n(n,t) = \mathrm{e}^{\mathrm{i}\theta_n}u_n(n,t), \tag{16}$$

where $\theta_n = pn + qt + \theta_0$. Then we have

$$\psi_{n+1}(n,t) = \mathrm{e}^{\mathrm{i}p+\mathrm{i}\theta_n}u_{n+1}(n,t), \tag{17}$$

$$\psi_{n-1}(n,t) = \mathrm{e}^{-\mathrm{i}p+\mathrm{i}\theta_n}u_{n-1}(n,t). \tag{18}$$

Substituting Eqs.(16)–(18) into Eq.(5) and setting the real and imaginary parts of the resulting equation equal to zero, one obtains

$$\begin{aligned}
&\mathrm{con}(p)(\alpha+\gamma u_n^2+\delta u_n^4)(u_{n+1}+u_{n-1})\\
&-(q+2\alpha)u_n+\beta u_n^3 = 0,
\end{aligned} \tag{19}$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + \sin(p)(\alpha+\gamma u_n^2+\gamma u_n^4)(u_{n+1}-u_{n-1}) = 0, \tag{20}$$

where $u_n \equiv u_n(n,t)$, $u_{n+1} \equiv u_{n+1}(n,t)$, $u_{n-1} \equiv u_{n-1}(n,t)$.

To look for the travelling wave solution of Eq.(5), we suppose $u_n = U(T_n), u_{n+1} = U_{n+1}(T_n), u_{n-1} = U_{n-1}(T_n)$, where $T_n = \tanh(\xi_n), \xi_n = kn + ct + \xi_0$. In this case, $p_s = 1$ or -1, $d_1 = k, c_1 = c$. Balancing the highest-order derivative term with that in Eq.(20), we obtain $m = 1$. We also have

$$U_n = a_0 + a_1T_n + b_1\sqrt{1-T_n^2}, \tag{21}$$

$$\begin{aligned}
U_{n+1} =&a_0 + a_1\frac{T_n+\tanh(k)}{1+T_n\tanh(k)}\\
&+ b_1\sqrt{1-\left[\frac{T_n+\tanh(k)}{1+T_n\tanh(k)}\right]^2},
\end{aligned} \tag{22}$$

$$\begin{aligned}
U_{n-1} =&a_0 + a_1\frac{T_n-\tanh(k)}{1-T_n\tanh(k)}\\
&+ b_1\sqrt{1-\left[\frac{T_n-\tanh(k)}{1-T_n\tanh(k)}\right]^2},
\end{aligned} \tag{23}$$

$$\frac{\mathrm{d}U_n}{\mathrm{d}t} = ca_1(1-T_n^2) - cb_1T_n\sqrt{1-T_n^2}\,. \tag{24}$$

Substituting Eqs.(21)–(24) into Eqs.(19) and (20), collecting all terms with the same order of T_n^i, $T_n^i\sqrt{1-T_n^2}$ $(i = 0,1,2,3,4)$ and setting the coefficients of each order of them to zero, yields a set of algebraic equations in $a_0, a_1, b_1, k, c, p, q$. To avoid tediousness, we omit the over-determined algebraic equations. With the aid of Maple 6, we obtain four

sets of solutions

$$
\begin{aligned}
p =&0, c=0, a_0=0, a_1=0,\\
q =&\frac{-\beta\gamma-4\delta\alpha\pm\beta\sqrt{\gamma^2-4\delta\alpha}}{2\delta},\\
k =&\text{arc cosh}\left(\frac{\beta(-\gamma\pm\sqrt{\gamma^2-4\delta\alpha})}{4\delta\alpha}\right),\\
b_1 =&\pm\frac{1}{2\delta\sqrt{2\alpha}}\\
&\times\sqrt{\gamma(\beta^2-4\delta\alpha)\mp(\beta^2+4\delta\alpha)\sqrt{\gamma^2-4\delta\alpha}}\ ;\\
p =&0, c=0, a_0=0, b_1=0,\\
q =&\frac{2\alpha(\beta+\gamma\mp\sqrt{\gamma^2-4\delta\alpha})}{-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}},\\
k =&\text{arc cosh}\left(\pm\sqrt{\frac{-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}{\beta}}\right),\\
a_1 =&\pm\sqrt{\frac{-\beta-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}{2\delta}};\\
p =&\pi, c=0, a_0=0, a_1=0,\\
q =&-\frac{\beta\gamma+4\delta\alpha\pm\beta\sqrt{\gamma^2-4\delta\alpha}}{2\delta},\\
k =&\text{arc cosh}\left(\frac{\beta(\gamma\pm\sqrt{\gamma^2-4\delta\alpha})}{4\delta\alpha}\right),\\
b_1 =&\pm\frac{1}{2\delta\sqrt{2\alpha}}\\
&\times\sqrt{\gamma(\beta^2-4\delta\alpha)\pm(\beta^2+4\delta\alpha)\sqrt{\gamma^2-4\delta\alpha}}\ ;\\
p =&\pi, c=0, a_0=0, b_1=0,\\
q =&\frac{-2\alpha(\beta+\gamma\pm\sqrt{\gamma^2-4\delta\alpha})}{\gamma\pm\sqrt{\gamma^2-4\delta\alpha}},\\
k =&\text{arc cosh}\left(\pm\sqrt{\frac{\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}{\beta}}\right),\\
a_1 =&\pm\sqrt{\frac{\beta-\gamma\mp\sqrt{\gamma^2-4\delta\alpha}}{2\delta}}. \qquad (25)
\end{aligned}
$$

According to Eqs.(16), (21) and (25), we obtain the following soliton solutions of Eq.(5)

Bright soliton solution

$$
\begin{aligned}
\psi_{n,1} =&\pm\frac{1}{2\delta\sqrt{2\alpha}}\\
&\times\sqrt{\gamma(\beta^2-4\delta\alpha)\mp(\beta^2+4\delta\alpha)\sqrt{\gamma^2-4\delta\alpha}}\\
&\times\text{sech}\left[n\text{arc cosh}\left(\frac{\beta(-\gamma\pm\sqrt{\gamma^2-4\delta\alpha})}{4\delta\alpha}\right)+\xi_0\right]\\
&\times\exp\left(\text{i}\frac{-\beta\gamma-4\delta\alpha\pm\beta\sqrt{\gamma^2-4\delta\alpha}}{2\delta}t+\text{i}\theta_0\right). \qquad (26)
\end{aligned}
$$

Here ξ_0 and θ_0 are arbitrary constants. α, γ and δ must require $\gamma^2-4\delta\alpha\geq 0$.

Dark soliton solution

$$
\begin{aligned}
\psi_{n,2} =&\pm\sqrt{\frac{-\beta-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}{2\delta}}\\
&\times\tanh\left[n\text{arc cosh}\left(\pm\sqrt{\frac{-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}{\beta}}\right)+\xi_0\right]\\
&\times\exp\left[\text{i}\frac{2\alpha(\beta+\gamma\mp\sqrt{\gamma^2-4\delta\alpha})}{-\gamma\pm\sqrt{\gamma^2-4\delta\alpha}}t+\text{i}\theta_0\right]. \qquad (27)
\end{aligned}
$$

The solutions (26) and (27) have been obtained by using standard Hirota method in Ref.[21]. The properties of the solutions (26) and (27) are shown in Figs.1 and 2.

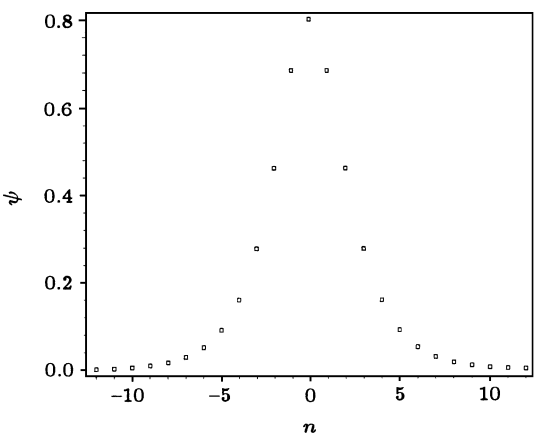


Fig.1. The bright soliton solution $\psi_{n,1}$ with $\alpha=2, \beta=0.4, \gamma=1, \delta=-0.1, \xi_0=0, \theta_0=0$ *at* $t=0$.

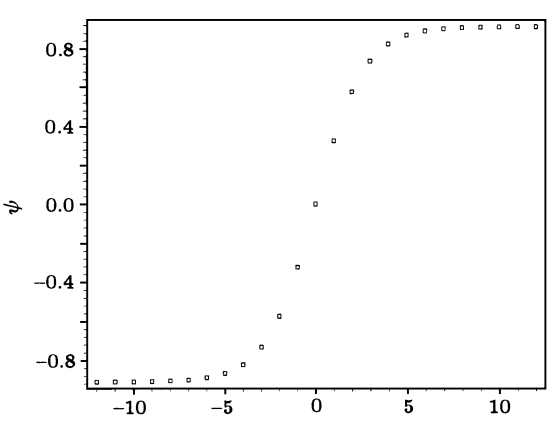


Fig.2. The dark soliton solution $\psi_{n,2}$ with $\alpha=-8, \beta=0.3, \gamma=1, \delta=0.025, \xi_0=0, \theta_0=0$ *at* $t=0$.

Alternating phase bright soliton

$$\psi_{n,3} = \pm \frac{(-1)^n}{2\delta\sqrt{2\alpha}} \times \sqrt{\gamma(\beta^2 - 4\delta\alpha) \pm (\beta^2 + 4\delta\alpha)\sqrt{\gamma^2 - 4\delta\alpha}} \times \operatorname{sech}\left[n\operatorname{arc\,cosh}\left(\frac{\beta(\gamma \pm \sqrt{\gamma^2 - 4\delta\alpha})}{4\delta\alpha}\right) + \xi_0\right] \times \exp\left[-\mathrm{i}\frac{\beta\gamma + 4\delta\alpha \pm \beta\sqrt{\gamma^2 - 4\delta\alpha}}{2\delta}t + \mathrm{i}\theta_0\right]. \quad (28)$$

Alternating phase dark soliton

$$\psi_{n,4} = \pm(-1)^n\sqrt{\frac{\beta - \gamma \mp \sqrt{\gamma^2 - 4\delta\alpha}}{2\delta}} \times \tanh\left[n\operatorname{arc\,cosh}\left(\pm\sqrt{\frac{\gamma \pm \sqrt{\gamma^2 - 4\delta\alpha}}{\beta}}\right) + \xi_0\right] \times \exp\left[\mathrm{i}\frac{-2\alpha(\beta + \gamma \pm \sqrt{\gamma^2 - 4\delta\alpha})}{\gamma \pm \sqrt{\gamma^2 - 4\delta\alpha}}t + \theta_0\right]. \quad (29)$$

The solutions (28) and (29) are obtained for the first time so far as we know. The properties of them are shown in Figs.3 and 4.

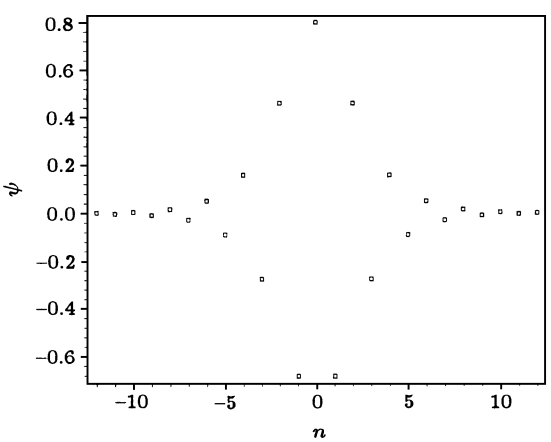


Fig.3. The alternating phase bright soliton solution $\psi_{n,3}$ with $\alpha = 2, \beta = -0.4, \gamma = 1, \delta = -0.1, \xi_0 = 0, \theta_0 = 0$ *at* $t = 0$.

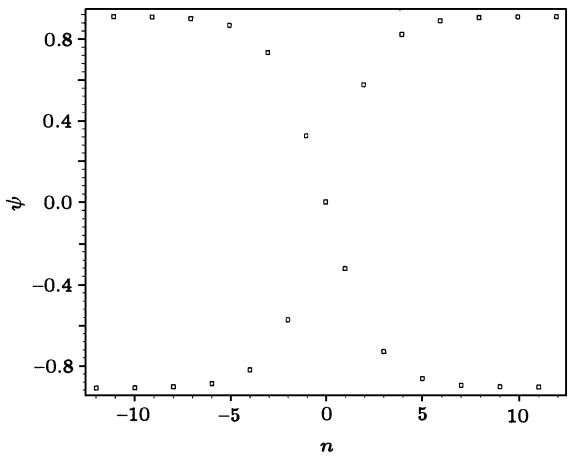


Fig.4. The alternating phase dark soliton solution $\psi_{n,4}$ with $\alpha = -8, \beta = -0.3, \gamma = 1, \delta = 0.025, \xi_0 = 0, \theta_0 = 0$ *at* $t = 0$.

4. Summary

By using the extended hyperbolic function approach, we have analysed the QDNLS equation. The QDNLS equation is not completely integrable, however the QDNLS equation has a number of exact localized solutions exists to the QDNLS equation provided that coefficients are bound by special relations. The set of localized solutions include bright soliton, dark soliton, alternating phase bright soliton and alternating phase dark soliton. The alternating phase bright soliton and alternating phase dark soliton solutions of the QDNLS equation are firstly obtained. Clearly these localized solutions represent only a small subset of large variety of possible solutions admitted by the QDNLS equation. we hope that they will useful in further perturbative and numerical analysis of various solutions to the QDNLS equation.

Acknowledgments

The authors would like to thank Professor Lou Sen-yue and Professor Huang Guo-xiang for helpful discussions.

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