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# A novel sliding mode nonlinear proportional-integral control scheme for controlling chaos

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A novel sliding mode nonlinear proportional-integral control (SMNPIC) scheme is proposed for driving a class of time-variant chaotic systems with uncertainty to arbitrarily desired trajectory with high accuracy. The SMNPIC differs from the previous sliding mode techniques in the sense that a nonlinear proportional-integral action of sliding function is involved in control law, so that both the steady-state error and the high-frequency chattering are reduced, and meanwhile, robustness and fastness are guaranteed. In addition, the proposed SMNPIC actually acts as a class of nonlinear proportional-integral-differential (PID) controller, in which the tracking error and its derivatives up to ($n$-1)th order as well as the integral of tracking error are considered, so that more useful information than traditional PID can be implemented and better dynamic and static characteristics can obtained. Its good performance for chaotic control is illustrated through a Duffing–Holmes system with uncertainty.

**Keywords:** chaos, sliding mode control, PI control
**PACC:** 0545

## 1. Introduction

Due to its powerful and potential applications in electronic devices and systems, laser systems, chemical reactions, bio-systems, power converters, information processing, neural networks, life and medical sciences, secure communications, etc, controlling those chaotic systems for engineering applications have emerged as an attractive and hot-line field in recent years, and many methods have been developed.[1−11] Thanks to its strong robustness, i.e. extreme insensitivity to system parameters and external disturbances, sliding mode control (SMC) seems to provide an effective solution to controlling chaos, and several SMC methods were introduced.[5,8−11] Unfortunately, a high-frequency chattering phenomenon exists in conventional SMC [12−15] which is very harmful to engineering application.[14,15] How to avoid or reduce the chattering is a key point for SMC. To date, several significant methods [13,16−18] have been proposed to eliminate or reduce its effects, such as a boundary-layer based method[16−18] and a turning reaching law based method.[13] Comparative analysis of those methods is beyond the scope of this paper.

In this paper, a novel sliding mode nonlinear proportional-integral control (SMNPIC) method is put forward, which is used for driving a class of time-variant chaotic systems with uncertainty to arbitrarily desired trajectory, even when the desired trajectories are not located on the embedded orbits of the chaotic systems. The SMNPIC differs from the previous sliding mode techniques in the sense that a nonlinear proportional-integral (PI) action of sliding function is involved in the control law. Since it is well known that PI control in general can reduce steady-state error, using the proposed controller will reduce the steady-state error. Moreover, the proposed control law is comprised of continuous functions instead of discontinuous ones. Consequently, both the steady-state error and the high-frequency chattering are reduced. At the same time, due to the special nonlinear functions involved, the SMNPIC is very insensitive to parameter uncertainty and external disturbances, and its speed of tracking desired trajectory is fast. Namely, both robustness and fastness are guaranteed in the SMNPIC proposed. As will be seen in the next sections, the proposed SMNPIC actually acts as a class of nonlinear proportional-integral-differential (PID) controller in which the tracking error and its derivatives up to ($n$-1)th order as well as the integral of tracking

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error are considered, so more useful information than from traditional PID can be implemented, and better dynamic and static characteristics can be obtained. Finally, a Duffing–Holmes system with uncertainty is taken as an illustrative example to validate the good performance of the proposed SMNPIC for controlling chaos.

## 2. Problem statement

Many chaotic systems can be described directly or after a diffeomorphism as follows.

$$\begin{aligned}&\dot{x}_i(t) = x_{i+1}(t), (1 \le i \le n-1),\\ &\dot{x}_n(t) = f(t,\boldsymbol{x}) + g(t,\boldsymbol{x})u(t) + \Delta f(t,\boldsymbol{x},u),\end{aligned} \quad (1)$$

where $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \Re^n$; $f(t,\boldsymbol{x})$, $g(t,\boldsymbol{x})$ are known continuous time-variant nonlinear functions and $g(t,\boldsymbol{x}) \neq 0$; $\Delta f(t,\boldsymbol{x},u)$ stands for the un-modelled parts, model perturbation and external disturbances and $|\Delta f(t,\boldsymbol{x},u)| < M$.

The goal of controlling chaos is to drive the chaotic system to track an $n$-dimensional desired vector $\boldsymbol{X}_{\mathrm{d}}$, which is defined as

$$\begin{aligned}\boldsymbol{X}_{\mathrm{d}} =&[x_{1\mathrm{d}}, x_{2\mathrm{d}}, \cdots, x_{n\mathrm{d}}]^{\mathrm{T}}\\ =&[x_{\mathrm{d}}(t), \dot{x}_{\mathrm{d}}(t), \cdots, x_{\mathrm{d}}^{(n-1)}(t)]^{\mathrm{T}} \in C[0,+\infty).\end{aligned}$$

Obviously, the control problem is changed into the original $n$th-order tracking problem of state.

Define the tracking error vector as

$$\begin{aligned}\boldsymbol{E}(t) =&\boldsymbol{X}_{\mathrm{d}} - \boldsymbol{x} = [x_{1d}(t) - x_1(t), \cdots, x_{nd}(t) - x_n(t)]^{\mathrm{T}}\\ =&[x_{\mathrm{d}}(t) - \dot{x}_1(t), \cdots, x_{\mathrm{d}}^{(n-1)}(t) - x_1^{(n-1)}(t)]^{\mathrm{T}}\\ =&[e(t), \dot{e}(t), \cdots, e^{(n-1)}(t)]^{\mathrm{T}}\\ =&[e_1(t), e_2(t), \cdots, e_n(t)]^{\mathrm{T}}.\end{aligned} \quad (2)$$

The control goal considered in this paper is that the proposed SMNPIC can drive the chaotic system to track an $n$-dimensional desired vector $\boldsymbol{X}_{\mathrm{d}}$ with high accuracy in a finite time, i.e. $\boldsymbol{E}(t)$ becomes arbitrarily small within the finite time. In other words,

$$||\boldsymbol{E}(t)|| \le \varepsilon, \quad \forall t \ge T^*, \quad (3)$$

where $T^*$ is a finite transient length and $\varepsilon$ is a sufficiently small positive number.

## 3. The SMNPIC law

The sliding function is described as follows:

$$s = e^{(n-1)} + c_1 e^{(n-2)} + \cdots + c_{n-1} e, \quad (4)$$

where $c_i$'s can be chosen according to the expected dynamic characteristic. Obviously, $s$=0 can be defined as sliding surface.

The proposed SMNPIC is then described as

$$\begin{aligned}u =&\Big[ - f(t,\boldsymbol{x}) + \sum_{i=1}^{n-1} c_i e^{(n-i)} + x_{\mathrm{d}}^{(n)} + r_1 s^{\frac{m_1}{n_1}} + r_2 s^{\frac{n_2}{m_2}}\\ &+ r_3 \Big( \int_0^t s\mathrm{d}t \Big)^{\frac{m_3}{n_3}} + r_4 \Big( \int_0^t s\mathrm{d}t \Big)^{\frac{n_4}{m_4}} \Big] / g(t,\boldsymbol{x}),\end{aligned} \quad (5)$$

where $r_i (i = 1, \cdots, 4) > M$, $m_i, n_i (i = 1, \cdots, 4)$ are odd numbers and $m_{2i-1} < n_{2i-1}, m_{2i} \le n_{2i} (i = 1, 2)$.

It is easily found that all functions in the SMNPIC are continuous instead of discontinuous, and a nonlinear PI action on the sliding function is involved. Consequently, both the steady-state error and the high-frequency chattering are reduced. Other good performances of the proposed SMNPIC will be analysed in the next sections.

## 4. The robustness analysis of the SMNPIC

**Lemma 1** Consider the system

$$\begin{aligned}\dot{z}_1(t) =&z_2(t),\\ \dot{z}_2(t) =& - r_3 z_1^{\frac{m_3}{n_3}}(t) - r_4 z_1^{\frac{n_4}{m_4}}(t)\\ &- r_1 z_2^{\frac{m_1}{n_1}}(t) - r_2 z_2^{\frac{n_2}{m_2}}(t).\end{aligned} \quad (6)$$

If $r_i (i = 1, \cdots, 4)$ are positive constants and $m_i, n_i (i = 1, \cdots, 4)$ are odds and $m_{2i-1} < n_{2i-1}, m_{2i} \le n_{2i} (i = 1, 2)$, the system (6) is asymptotically stable at (0,0).

**Proof** Choose a Lyapunov function as

$$V(z_1, z_2) = \frac{2r_3 n_3}{m_3 + n_3} z_1^{\frac{m_3+n_3}{n_3}} + \frac{2r_4 m_4}{m_4 + n_4} z_1^{\frac{m_4+n_4}{m_4}} + z_2^2, \quad (7)$$

then, differentiating $V$ with respect to $t$ and using Eq.(6) yield

$$\begin{aligned}\dot{V} =&(2r_3 z_1^{\frac{m_3}{n_3}} + 2r_4 z_1^{\frac{n_4}{m_4}}) z_2\\ &+ 2z_2(-r_3 z_1^{\frac{m_3}{n_3}} - r_4 z_1^{\frac{n_4}{m_4}} - r_1 z_2^{\frac{m_1}{n_1}} - r_2 z_2^{\frac{n_2}{m_2}})\\ =&-2(r_1 z_2^{\frac{m_1+n_1}{n_1}} + r_2 z_2^{\frac{n_2+m_2}{m_2}}) \le 0.\end{aligned} \quad (8)$$

Based on Lyapunov's stability theory, it is easily found that the system is asymptotically stable at (0,0).

**Theorem 1**. For the system (1), if the SMNPIC law (5) is adopted and $n_1/m_1$ is chosen to be big enough, then the system can approach a sufficiently small neighbourhood $\Delta$ of the sliding surface $s$=0 in

a finite time,
where $\Delta := \{\boldsymbol{x} : |s| \le (M/r_1)^{n_1/m_1}\}$.

**Proof** Substituting Eq.(5) into Eq.(1), we have

$$\begin{aligned} x_1^{(n)} =& \sum_{i=1}^{n-1} c_i e^{(n-i)} + x_{\mathrm{d}}^{(n)} + r_1 s^{\frac{m_1}{n_1}} + r_2 s^{\frac{n_2}{m_2}} \\ &+ r_3 \left( \int_0^t s \mathrm{d}t \right)^{\frac{m_3}{n_3}} \\ &+ r_4 \left( \int_0^t s \mathrm{d}t \right)^{\frac{n_4}{m_4}} + \Delta f(t, \boldsymbol{x}, u). \end{aligned} \tag{9}$$

From Eqs.(4) and (2), we have

$$\begin{aligned} \dot{s} =& e^{(n)} + c_1 e^{(n-1)} + \cdots + c_{n-1} \dot{e} \\ =& x_{\mathrm{d}}^{(n)} - x_1^{(n)} + c_1 e^{(n-1)} + \cdots + c_{n-1} \dot{e} \, . \end{aligned} \tag{10}$$

Substituting Eq.(9) into Eq.(10) yields

$$\begin{aligned} &\dot{s} + r_1 s^{\frac{m_1}{n_1}} + r_2 s^{\frac{n_2}{m_2}} + r_3 \left( \int_0^t s \mathrm{d}t \right)^{\frac{m_3}{n_3}} \\ &+ r_4 \left( \int_0^t s \mathrm{d}t \right)^{\frac{n_4}{m_4}} \\ =& - \Delta f(t, \boldsymbol{x}, u). \end{aligned} \tag{11}$$

If $\boldsymbol{x} \in \{\boldsymbol{x} : \ |s| > (M/r_1)^{n_1/m_1}\}$, then

$$r_1 > \frac{M}{|s|^{m_1/n_1}}. \tag{12}$$

Thus, let

$$r_1 = \frac{M}{|s|^{m_1/n_1}} + \delta, \tag{13}$$

where $\delta > 0$.
From Eq.(11), we have

$$\begin{aligned} &\dot{s} + \left( r_1 + \frac{\Delta f(t, \boldsymbol{x}, u)}{s^{m_1/n_1}} \right) s^{\frac{m_1}{n_1}} + r_2 s^{\frac{n_2}{m_2}} \\ &+ r_3 \left( \int_0^t s \mathrm{d}t \right)^{\frac{m_3}{n_3}} \\ &+ r_4 \left( \int_0^t s \mathrm{d}t \right)^{\frac{n_4}{m_4}} = 0, \end{aligned} \tag{14}$$

i.e.,

$$\begin{aligned} &\dot{s} + \bar{r}_1 s^{\frac{m_1}{n_1}} + r_2 s^{\frac{n_2}{m_2}} + r_3 \left( \int_0^t s \mathrm{d}t \right)^{\frac{m_3}{n_3}} \\ &+ r_4 \left( \int_0^t s \mathrm{d}t \right)^{\frac{n_4}{m_4}} = 0, \end{aligned} \tag{15}$$

where

$$\bar{r}_1 = r_1 + \frac{\Delta f(t, \boldsymbol{x}, u)}{s^{m_1/n_1}}. \tag{16}$$

Thus, substituting Eq.(13) into Eq.(16), we obtain

$$\begin{aligned} \bar{r}_1 =& r_1 + \frac{\Delta f(t, \boldsymbol{x}, u)}{s^{m_1/n_1}} \\ =& \frac{M}{|s|^{m_1/n_1}} + \delta + \frac{\Delta f(t, \boldsymbol{x}, u)}{s^{m_1/n_1}} \\ \ge& \frac{M}{|s|^{m_1/n_1}} + \delta - \frac{|\Delta f(t, \boldsymbol{x}, u)|}{|s|^{m_1/n_1}} \ge \delta > 0. \end{aligned} \tag{17}$$

For Eq.(15), let $z_1(t) = \int_0^t s \mathrm{d}t, z_2(t) = s$, then Eq.(15) can be easily rewritten as

$$\begin{aligned} \dot{z}_1(t) =& z_2(t), \\ \dot{z}_2(t) =& - r_3 z_1^{\frac{m_3}{n_3}}(t) - r_4 z_1^{\frac{n_4}{m_4}}(t) \\ &- \bar{r}_1 z_2^{\frac{m_1}{n_1}}(t) - r_2 z_2^{\frac{n_2}{m_2}}(t). \end{aligned} \tag{18}$$

From Lemma 1 and the existence condition of Eq.(12), it is easily known that Eq.(18) can approach the neighbourhood $\Delta$ of the sliding surface $s = 0$ in a finite time.

Obviously, if $r_1 > M$ and $n_1/m_1$ are chosen to be big enough, we can make $(M/r_1)^{n_1/m_1}$ sufficiently small. This means that $\Delta$ is sufficiently small.

**Remark 1** From Theorem 1, it is obvious that, if $r_1$ and $n_1/m_1$ are appropriately chosen, the proposed SMNPIC can make $\Delta$ sufficiently small and then is quite effective. For example, in the case of $M = 1$, choosing $r_1 = 3$ and $n_1/m_1 = 7$ can make $\Delta = \{\boldsymbol{x} : \ |s| \le 4.57 \times 10^{-4}\}$. Of course, we can also design the accurate size of $\Delta$ beforehand. For example, if we want to design a sufficiently small positive number $\delta$ such that $\Delta = \{\boldsymbol{x} : \ |s| \le \delta\}$, we only need choosing $r_1$ and $n_1/m_1$ such that $(M/r_1)^{n_1/m_1} = \delta$.

**Remark 2** From the proof of Theorem 1, it is easily found that the existence condition of Theorem 1 is that $r_1 > M$ and $n_1/m_1$ is sufficiently big; and other parameters in Eq.(5) need not be restricted additionally. As a result, to simplify the controller design and without loss of generality, the parameters of control law (5) can be chosen as

$$\begin{aligned} &r_1 = r_2 = r_3 = r_4 > M, \\ &\frac{m_1}{n_1} = \frac{m_3}{n_3} < 1, \ \frac{n_2}{m_2} = \frac{n_4}{m_4} \ge 1. \end{aligned}$$

**Lemma 2** Consider the system

$$\dot{\boldsymbol{\phi}}(t) = \boldsymbol{A}\boldsymbol{\phi}(t) + \boldsymbol{\theta}(t), \tag{19}$$

where $\boldsymbol{\phi}(t), \boldsymbol{\theta}(t) \in \Re^n$, $\boldsymbol{A}$ is an $n \times n$ constant square matrix.

If the system (19) satisfies the following conditions: a) $\boldsymbol{A}$ is stable, and b) $||\boldsymbol{\theta}(t)|| \le L$, then

$$||\boldsymbol{\phi}(t)|| \le \eta L/\alpha, \tag{20}$$

where $\eta > 0, \alpha \geq -\max\limits_{i=1,\cdots,n} \mathrm{Re}[\lambda_i(\boldsymbol{A})]$ with $\mathrm{Re}[\lambda_i(\boldsymbol{A})]$ denoting the real part of eigenvalues of $\boldsymbol{A}$.

**Proof** See Appendix.

**Theorem 2** For the system (1), if the SMNPIC law (5) is adopted and $n_1/m_1$ is chosen to be big enough, then Eq.(3) can be satisfied.

**Proof** Obviously, $c_i$ in Eq.(4) can easily be chosen to make matrix $\bar{\boldsymbol{A}}$ stable, where $\bar{\boldsymbol{A}}$ is defined as

$$\bar{\boldsymbol{A}} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -c_{n-1} & -c_{n-2} & -c_{n-3} & \cdots & -c_1 \end{bmatrix}. \quad (21)$$

Define

$$\tilde{\boldsymbol{E}}(t) = [e(t), \dot{e}(t), \cdots, e^{(n-2)}(t)]^{\mathrm{T}}, \quad (22a)$$

and

$$\tilde{\boldsymbol{s}}(t) = [\underbrace{0, 0, \cdots 0}_{n-2}, s(t)]^{\mathrm{T}}. \quad (22b)$$

From Eqs.(4) and (20), we have

$$\dot{\tilde{\boldsymbol{E}}}(t) = \bar{\boldsymbol{A}}\tilde{\boldsymbol{E}}(t) + \tilde{\boldsymbol{s}}(t). \quad (23)$$

From Theorem 1, there exists $t^* > 0$ such that

$$|s(t)| \leq (M/r_1)^{n_1/m_1}, \quad \forall t \geq t^*. \quad (24)$$

Applying Lemma 2 in Eqs.(23) and (24) yields

$$||\tilde{\boldsymbol{E}}(t)|| \leq \kappa(M/r_1)^{n_1/m_1}/\bar{\alpha}, \quad \forall t \geq t^*, \quad (25)$$

where $\kappa > 0, \bar{\alpha} \geq -\max\limits_{i=1,\cdots,n} \mathrm{Re}[\lambda_i(\bar{\boldsymbol{A}})]$. From Eqs.(2), (22) and (4), we obtain

$$\boldsymbol{E}(t) = \begin{bmatrix} \tilde{\boldsymbol{E}}(t) \\ e^{(n-1)} \end{bmatrix} \quad (26a)$$

and

$$|e^{(n-1)}| \leq ||\bar{\boldsymbol{A}}\tilde{\boldsymbol{E}}(t) + \tilde{\boldsymbol{s}}(t)||, \quad (26b)$$

then

$$\begin{aligned} ||\boldsymbol{E}(t)|| \leq & ||\tilde{\boldsymbol{E}}(t)|| + ||\bar{\boldsymbol{A}}\tilde{\boldsymbol{E}}(t) + \tilde{\boldsymbol{s}}(t)|| \\ \leq & (1 + ||\bar{\boldsymbol{A}}||)||\tilde{\boldsymbol{E}}(t)|| + |s(t)|. \end{aligned} \quad (27)$$

From Eqs.(24), (25) and (27), we have

$$||\boldsymbol{E}(t)|| \leq \tilde{\kappa}(M/r_1)^{n_1/m_1} \quad \forall t \geq t^*, \quad (28)$$

where $\tilde{\kappa} = 1 + (1 + ||\bar{\boldsymbol{A}}||)\kappa/\bar{\alpha}$.

Thus, if the value of $\tilde{\kappa}(M/r_1)^{n_1/m_1}$ is small enough, the arbitrarily desired tracking accuracy can be achieved and the control goal (3) is realized. In fact, if choosing sufficiently large $r_1$ and $n_1/m_1$, we can make the value of $\tilde{\kappa}(M/r_1)^{n_1/m_1}$ small enough, which has been clearly explained in Remark 1.

## 5. The fastness analysis of the SMNPIC

It is well known that the motion of a system can be broken into two phases. Firstly, the system approaches the sliding surface until it reaches the sufficiently small neighbourhood of the sliding surface; then it moves approximately along the sliding surface. Consequently, the speed of convergence of the SMNPIC is determined by that of the system (6) as well as the characteristic equation of sliding surface. Because the characteristic equation of sliding surface can be arbitrarily designed, we only analyse the speed of convergence of the system (6). The phase plane and the state responses can be taken as two index marks to display the speed of convergence of the system (6). The parameters in the system (6) are chosen as $r_i = 1, m_i = 3, n_i = 5 (1 \leq i \leq 4)$. Due to the parity of the system (6), we only need studying the initial point located in the first and the fourth quadrants. The results are shown in Figs.1–4.

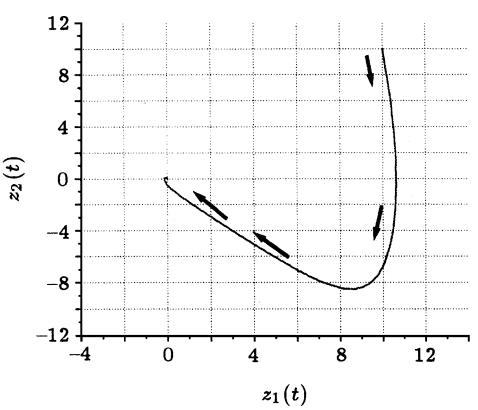



**Fig.1.** The phase plane for initial point (10,10).

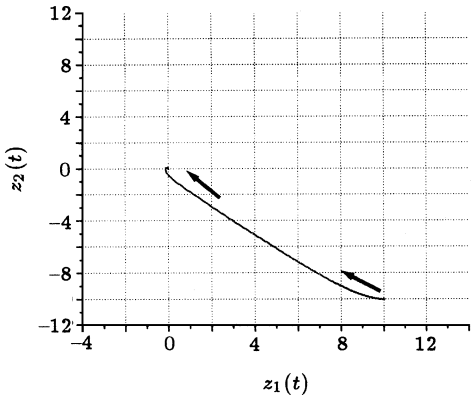



**Fig.2.** The phase plane for initial point (10,-10).

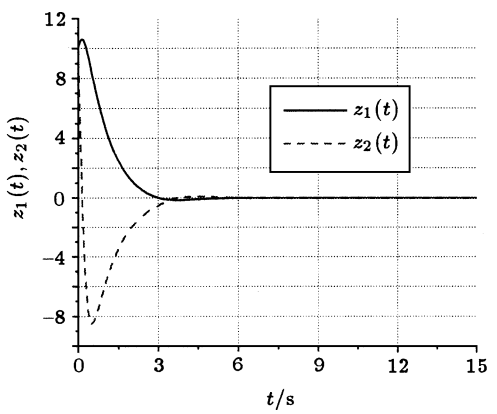



**Fig.3.** The state responses for initial condition (10,10).

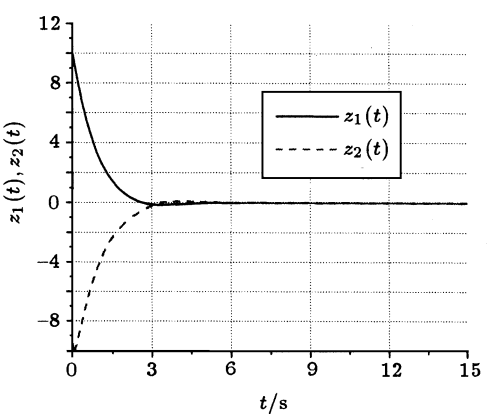



**Fig.4.** The state responses for initial condition (10,-10).

Figures 1 and 2 illustrate the phase plane $z_2(t)$ against $z_1(t)$ for the initial points located in the first and fourth quadrants, respectively. Figures 3 and 4 show the time response of $z_1(t)$ and $z_2(t)$ for the initial points located in the first and fourth quadrants, respectively.

From the results shown in Figs.1–4, it can be found that the state of the system (6) can rapidly approach the equilibrium point (0,0), i.e. the proposed SMNPIC can approach the sliding surface quickly. If the parameters in the system (6) are appropriately chosen, the speed of convergence of the system (6) can be designed to be fast enough, so that the two phases of the system motion can be sufficiently fast as expected. Consequently, the tracking error can be arbitrarily small desired in a finite time which can also be reduced to be small enough. It implies that both accuracy and fastness of the SMNPIC are simultaneously guaranteed.

# 6. The relationship with PID

**Case 1** The controlled system is a second-order system and $f(t,\boldsymbol{x})$ and $x_{\rm d}^{(n)}$ are considered in uncertainty $\Delta f(t,\boldsymbol{x},u)$, $g(t,\boldsymbol{x})\equiv b$ ($b$ is a constant number and nonzero. Without loss of generality, we choose $b=1$). Accordingly, the actual SMNPIC law is

$$\begin{aligned} u =& c_1\dot{e}+r_1(\dot{e}+c_1e)^{\frac{m_1}{n_1}}+r_2(\dot{e}+c_1e)^{\frac{n_2}{m_2}} \\ &+r_3\left(e+c_1\int_0^t e{\rm d}t\right)^{\frac{m_3}{n_3}} \\ &+r_4\left(e+c_1\int_0^t e{\rm d}t\right)^{\frac{n_4}{m_4}}. \end{aligned} \tag{29}$$

Obviously, Eq.(29) is a nonlinear PID controller.

Similarly, for a high-order system, the proposed control law actually is a class of nonlinear PID. In such a circumstance, the tracking error and its derivatives up to ($n$-1)th order as well as the integral of tracking error are considered in the controller, so more useful information than from traditional PID can be implemented. Consequently, applying the proposed controller can obtain better dynamic and static characteristics.

# 7. Illustrative example

Consider a Duffing–Holmes oscillator

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p_1x_1-p_2x_2-x_1^3+q\cos(w_1t)+\Delta f(t,\boldsymbol{x},u)+u. \end{aligned} \tag{30}$$

The forthcoming simulation results will show that once SMNPIC is adopted in Eq.(30), the chaotic system can be driven to an arbitrarily desired trajectory, even when the desired trajectory is not located on the embedded orbits of the chaotic system.

Obviously, the corresponding nominal system of Eq.(30) is described as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= p_1x_1-p_2x_2-x_1^3+q\cos(w_1t). \end{aligned} \tag{31}$$

Choosing $p_1 = 1$, $p_2 = 0.25$, $q = 0.3$, $w_1 = 1$ yields that the nominal system (31) displays chaotic behaviour[8] as plotted in Fig.5.

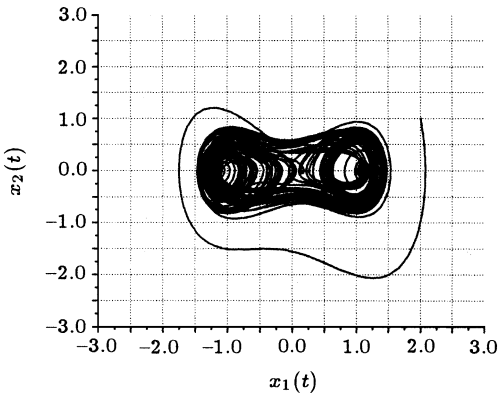



**Fig.5.** Chaotic behaviour of the nominal system (32) for an initial state (2,1).

Let the desired trajectory be

$$x_{\mathrm{d}}(t) = A\sin(w_2 t). \tag{32}$$

If the parameters in Eq.(32) are $A = 1$, $w_2 = 1.2$, then it is obvious that the desired trajectory $\sin(1.2t)$ is not one of the embedded orbits of the strange attractor.

Let the initial values of the state of the system (30) be $(x_1(0), x_2(0)) = (2, 1)$ and the sliding surface is designed as

$$s = \dot{e} + e. \tag{33}$$

Choose $r_1 = r_2 = r_3 = r_4$=20,$n_1 = n_3$=3 and $m_1 = m_2 = m_3 = m_4 = n_2 = n_4$=1, then the actual control law can be described as

$$\begin{aligned} u = &- p_1 x_1 + p_2 x_2 + x_1^3 - q\cos(w_1 t) - 1.44\sin(1.2t) \\ &+ 20s^{1/3} + 20s + 20\left(\int_0^t s\mathrm{d}t\right)^{1/3} + 20\int_0^t s\mathrm{d}t, \end{aligned} \tag{34}$$

where $s$ has been defined in Eq.(33).

To display the robustness of the proposed SMNPIC, we make $p_1$, $p_2$ double and take $2\sin(2t)\exp(-0.1t)$ as the external disturbance, i.e.

$$\Delta f(t, \boldsymbol{x}, u) = 2\sin(2t)\exp(-0.1t) + x_1 - 0.25x_2 \tag{35}$$

is considered as uncertainty in Eq.(30).

The simulated results are plotted in Figs.6–11. Figures 6 and 7 show that the sufficiently small neighbourhood of the sliding surface is approached in a finite time. For examples, the size of the neighbourhood is smaller than $10^{-3}$ after 0.38 s; $10^{-4}$ after 0.63s. The following motion of the system is forced to run according to the dynamic characteristic equation $\dot{e} + e = 0$ and actually is validated by the simulated results as shown in Figs.8–10. Since all the functions in SMNPIC are continuous, the chattering is reduced. In fact, the chattering reduction is validated by the simulated result plotted in Fig.11.

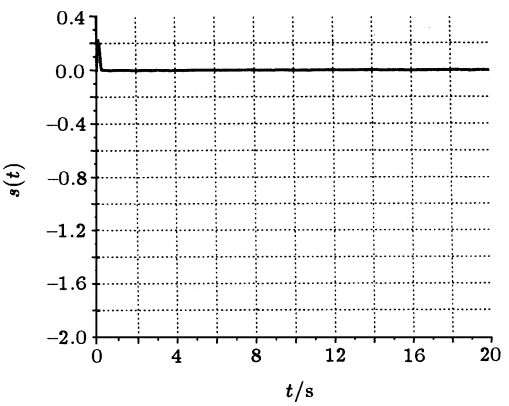



**Fig.6.** Time response of $s(t)$.

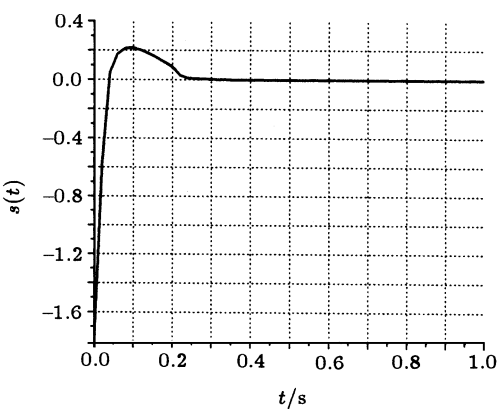



**Fig.7.** The local augment of Fig.6.

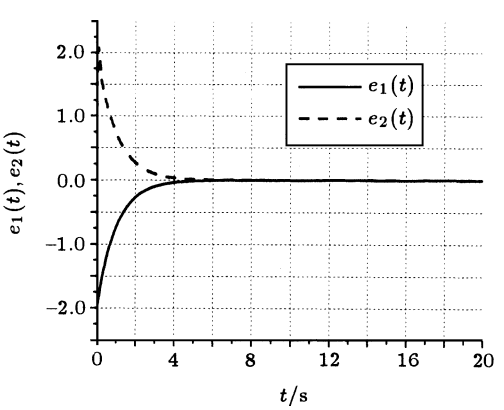



**Fig.8.** Time response of the tracking error vector.

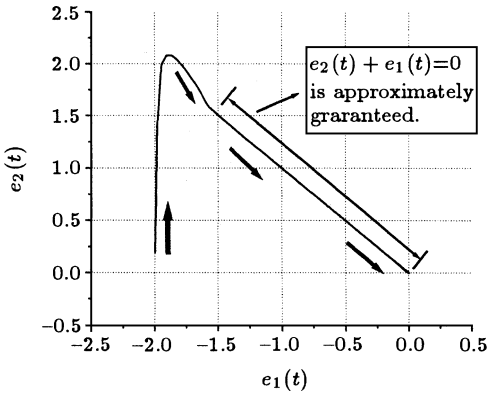



**Fig.9.** The phase plane of $e_1$ versus $e_2$; $e_2(t)+e_1(t) = 0$ is approximately guaranteed in a finite time.

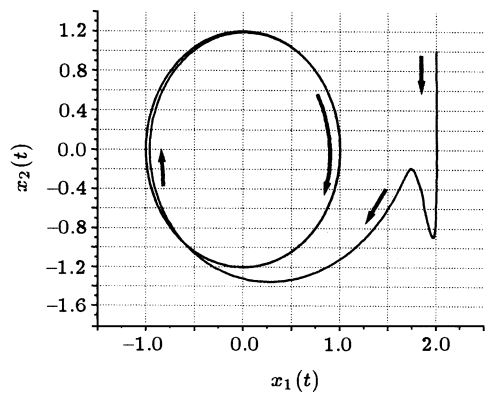



**Fig.10.** The phase plane of $x_1$ versus $x_2$; $[x_1(t)]^2 + [x_2(t)/1.2]^2 = 1$ is approximately guaranteed in a finite time.

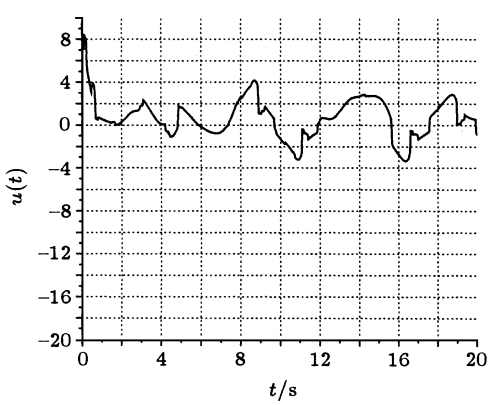



**Fig.11.** The controller action of SMNPIC.

## 8. Conclusions

A novel SMNPIC method is proposed for driving a class of time-variant chaotic systems with uncertainty to an arbitrarily desired trajectory with high accuracy. Due to nonlinear PI action of sliding function in the SMNPIC, both the steady-state error and the high-frequency chattering are reduced, and meanwhile, robustness and fastness are guaranteed. In addition, the SMNPIC actually acts as a class of nonlinear PID controller and can obtain good dynamic and static characteristics. The good performance of the proposed SMNPIC for chaotic control is presented through a Duffing–Holmes system with uncertainty. The proposed SMNPIC can also be applied in synchronization of chaos, since the problem can be changed into the original $n$th-order tracking problem of state. In brief, both theoretic and illustrative results display that the SMNPIC scheme provides a promising solution to the control and synchronization of chaos.

## Appendix

**Proof of Lemma 2**

**Proof** Based on the linear system theory, the only solution of Eq.(19) is

$$\boldsymbol{\varphi}(t) = \int_{-\infty}^{t} \mathrm{e}^{A(t-\tau)}\boldsymbol{\theta}(t)d\tau. \tag{A1}$$

Since $\boldsymbol{A}$ is an $n \times n$ constant stable square matrix, the following inequality can be easily derived and also can be found in Ref.[19] and its references therein.

$$||\mathrm{e}^{A(t-\tau))}|| \leq \eta \mathrm{e}^{-\alpha(t-\tau)}, \tag{A2}$$

where $\eta > 0, \alpha \geq -\max\limits_{i=1,\cdots,n} \mathrm{Re}[\lambda_i(\boldsymbol{A})]$.

From Eqs.(A1) and (A2), we have

$$||\boldsymbol{\varphi}(t)|| \leq \int_{-\infty}^{t} \eta \mathrm{e}^{-\alpha(t-\tau)}||\boldsymbol{\theta}(t)||\mathrm{d}\tau. \tag{A3}$$

For Eq.(A3), using the condition b) shown in Section 4 yields

$$\begin{aligned}||\boldsymbol{\varphi}(t)|| \leq & \int_{-\infty}^{t} \eta \mathrm{e}^{-\alpha(t-\tau)}||\boldsymbol{\theta}(t)||\mathrm{d}\tau \\ \leq & \eta L \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha\tau}\mathrm{d}\tau = \eta L/\alpha. \quad \text{QED.}\end{aligned}$$

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