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Chaos and chaotic control in a fractional-order electronic oscillator

Gao Xin(高 心)[a)b)†] and Yu Jue-Bang(虞厥邦)[a)]

[a)] *School of Electronic Engineering, University of Electronics Science and Technology of China, Chengdu* 610054, *China*
[b)] *School of Electrical and Information Engineering, Southwest University for Nationalities of China, Chengdu* 610041, *China*

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In this paper, we study the chaotic behaviours in a fractional-order chaotic electronic oscillator. We find that chaos exists in the fractional-order electronic oscillator with an order being less than 3. In addition, we numerically simulate the continuance of the chaotic behaviours in the electronic oscillator with orders ranging from 2.8 to 3.2. Finally, we further investigate the method of controlling a fractional-order electronic oscillator based on adaptive backstepping. Numerical simulations show the effectiveness and feasibility of this approach.

Keywords: fractional order, chaos, chaotic control, adaptive backstepping
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1. Introduction

Chaotic systems have held researchers' interest in the past few decades. Such nonlinear systems can occur in various natural and man-made systems, and are known to have great sensitivity to initial conditions. Thus, two trajectories starting from their arbitrary and almost the same initial states in such systems could evolve in dramatically different fashions, and soon become uncorrelated and unpredictable. In recent years, a new direction of chaos research has emerged, in which fractional-order calculus is applied to dynamic systems.

Fractional calculus is a 300-year-old topic, but its application to physics and engineering is just a recent focus of interest. Many systems are known to display fractional-order dynamics, such as viscoelastic systems,[1] electrode–electrolyte polarization,[2] and electromagnetic waves.[3] More recently, there appears a new trend in the investigation of the control and dynamics of fractional-order dynamical systems, such as Chua's system,[4] Wien bridge oscillator,[5] "jerk" model,[6] etc. Ahmad and Sprott[6] have presented a conjecture that the third-order chaotic nonlinear system can still produce chaotic behaviour with a total system order of $2+\varepsilon$, where $1 > \varepsilon > 0$, using the appropriate control parameters. In Ref.[4], it has been shown that the fractional-order Chua circuit which an order being as low as 2.7 can produce a chaotic attractor. In Ref.[6], chaotic behaviours of the fractional-order "jerk" model has been studied, in which chaotic attractor is generated with a system order being as low as 2.1. And chaotic control[7] and synchronization[8] of fractional-order chaotic systems are also investigated. In this paper, we study the chaotic behaviours in a fractional-order electronic oscillator. We find that chaos exists in the fractional-order electronic oscillator with an order being less than 3. In addition, we numerically simulate the continuance of the chaotic behaviours in the electronic oscillator with orders from 2.8 to 3.2. Finally, in this paper, we further investigate the method in controlling the fractional-order chaotic electronic oscillator based on adaptive backstepping. The backstepping[9,10] is a recursive procedure for systematically selecting the control of Lyapunov functions (clf) that allows the design of adaptive controllers for a class of nonlinear process. This method can solve the singularity problem.[11] Numerical simulations show the effectiveness and feasibility of this approach.

2. Fractional derivative and its approximation

†Correspondence author: E-mail: gxin7993@163.com

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the first reference probably associated with Leibnitz in 1695. Two commonly used definitions for the general fractional differintegral are the Grunwald–Letnikov (GL) definition and the Riemann–Liouviller (RL) definition.[12] The RL definition is given here by

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \mathrm{d}\tau, \qquad (1)$$

where $n-1 \le q < n$ and $\Gamma(\bullet)$ is the Euler's gamma function. Considering all the initial conditions to be zero, the Laplace transform of the RL fractional derivative is given by

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\}. \qquad (2)$$

Thus, the fractional integral operator of an order of "q" can be represented by the transfer function $F(s) = 1/s^q$ in the frequency domain.

The standard definitions of fractional diff-integral do not allow direct implementation of the operator in time-domain simulations. An effective method in dealing with this problem is to approximate fractional operators by using the standard integer-order operators. The approximate approach taken here is the one presented in Ref.[13]. Basically, the idea is to approximate the system behaviour in the frequency domain, using a specified error in decibels and a bandwidth to generate a continuous sequence of pole-zero pairs for the system with a single fractional power pole. This approximation is created by choosing an initial break-point, the line with a slop of $-20q$dB/decade is approximated by a number of zig–zag lines connected together with alternative slops of 0 dB/decade and -20dB/decade. Thus an approximation of any desired accuracy over any frequency band can be achieved. In Table 1 of Ref.[6], approximations for $1/s^q$ with $q = 0.1 - 0.9$ in steps being 0.1 have been given with errors of approximately 2dB. We mainly use these approximations in the following simulations.

3. The fractional-order chaotic electronic oscillator model

Here, we consider a chaotic electronic oscillator model with a control parameter. It has been reported in Ref.[14] that the standard derivative is replaced by a fractional derivative as follows:

$$\begin{aligned} \frac{\mathrm{d}^\alpha x_1}{\mathrm{d}t^\alpha} &= x_2, \\ \frac{\mathrm{d}^\beta x_2}{\mathrm{d}t^\beta} &= x_3, \\ \frac{\mathrm{d}^\gamma x_3}{\mathrm{d}t^\gamma} &= -a[x_3 + x_2 + x_1 - f(x_1)], \end{aligned} \qquad (3)$$

where the model parameter a is allowed to be varied, and α, β, γ are the fractional orders. $f(x_1) = \mathrm{sgn}(x_1)$ represents the model nonlinearity. When $\alpha = \beta = \gamma = 1$, Eq.(3) is equivalent to that for an original integer-order electronic oscillator, which is chaotic when $0.49 < a < 1.0$.

Simulations are performed for total mathematical system order being 2.8, 2.9, 3, 3.1, and 3.2 respectively. The phase portraits are shown in Figs.1–5, respectively.

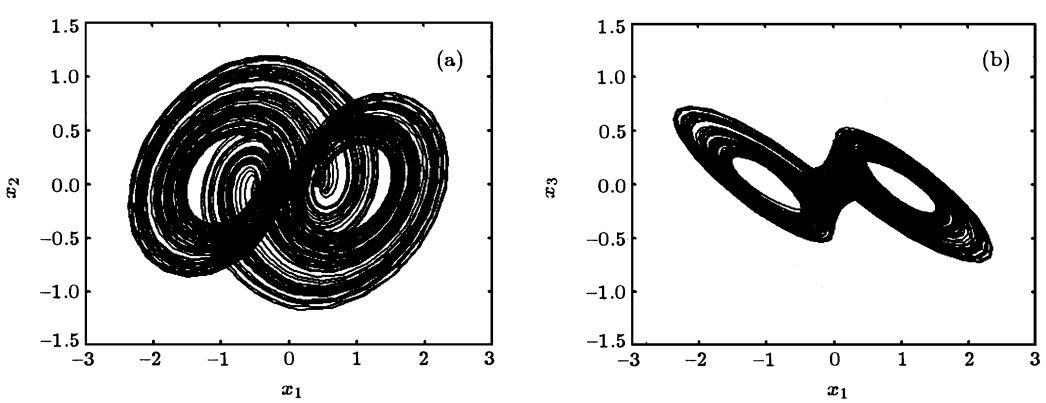


Fig.1. Phase plane portraits for the fractional-order electronic oscillator model: (a) x_2 vs x_1, total mathematical system order is 2.8, a=0.2, α=0.9, β=0.9, γ=1, (b) x_3 vs x_1, total mathematical system order is 2.8, a=0.2, α=0.9, β=0.9, γ=1.

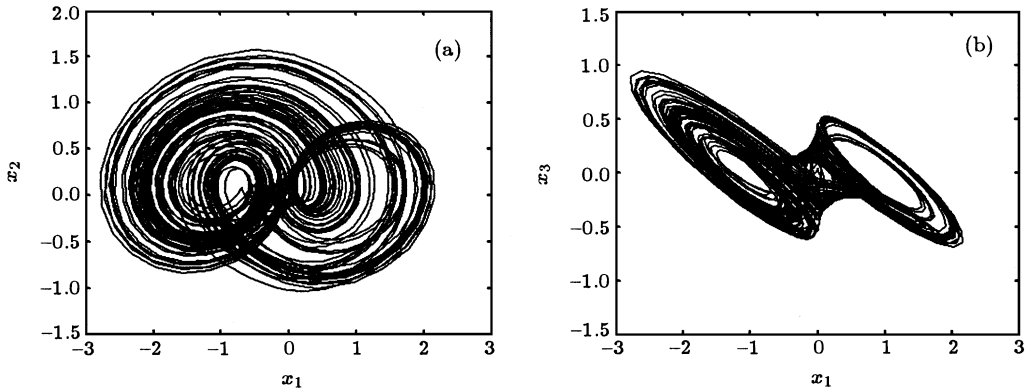


Fig.2. Phase plane portraits for the fractional-order electronic oscillator model: (a) x_2 vs x_1, total mathematical system order is 2.9, a=0.32, α=1, β=0.9, γ=1, (b) x_3 vs x_1, total mathematical system order is 2.9, a=0.32, α=1, β=0.9, γ=1.

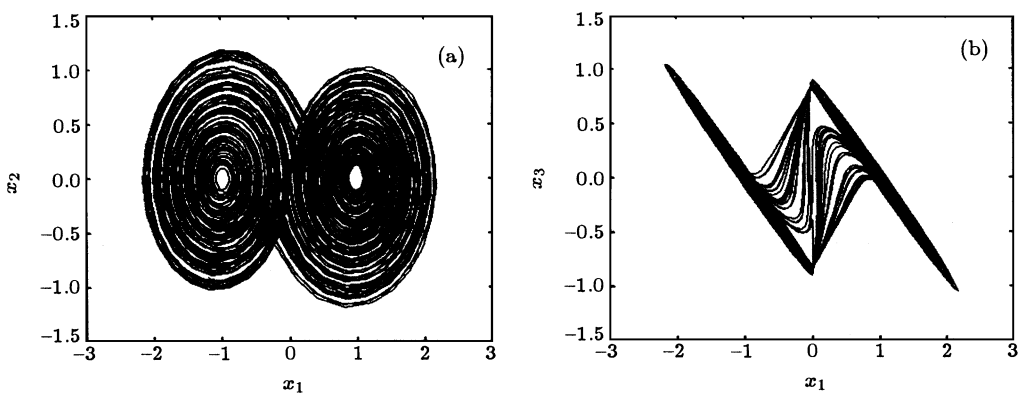


Fig.3. Phase plane portraits for the electronic oscillator model: (a) x_2 vs x_1, total mathematical system order is 3, a=0.8, $\alpha = \beta = \gamma = 1$, (b) x_3 vs x_1, total mathematical system order is 3, a=0.8, $\alpha = \beta = \gamma = 1$.

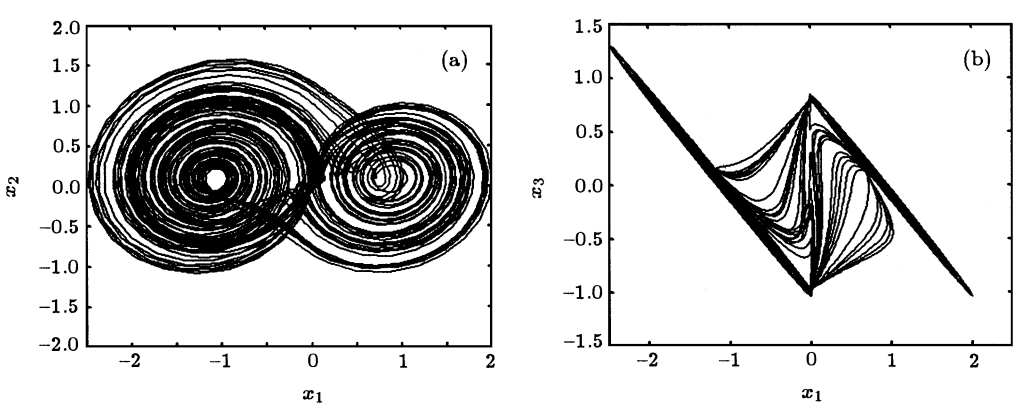


Fig.4. Phase plane portraits for the fractional-order electronic oscillator model: (a) x_2 vs x_1, total mathematical system order is 3.1, a=0.9, α=1.1, $\beta = \gamma = 1$, (b) x_2 vs x_1, total mathematical system order is 3.1, a=0.9, α=1.1, $\beta = \gamma = 1$.

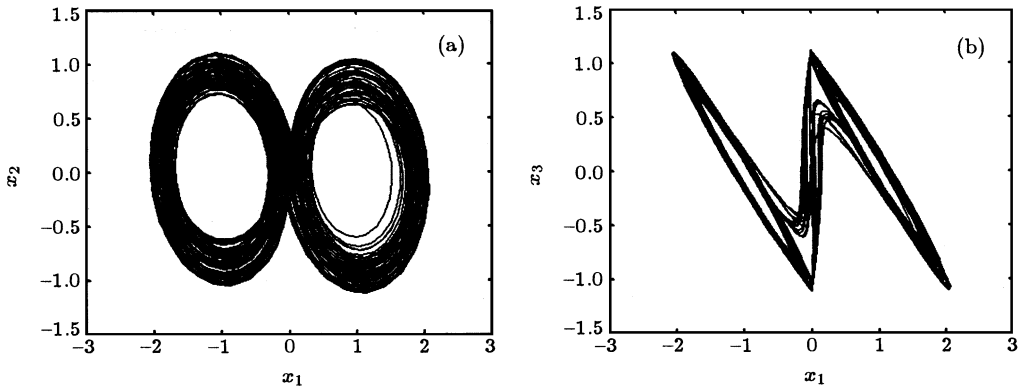


Fig.5. Phase plane portraits for the fractional-order electronic oscillator model: (a) x_2 vs x_1, total mathematical system order is 3.2, a =1.18, $\alpha = \beta = 1.1, \gamma = 1$, (b) x_3 vs x_1, total mathematical system order is 3.2, a= 1.18, $\alpha = \beta = 1.1, \gamma = 1$.

The simulation results demonstrate that the chaos indeed exists in the fractional-order electronic oscillator with an order being less than 3. We can see that the chaotic attractors of the fractional-order electronic oscillator are similar to those of the integer-order electronic oscillator. The continuance of the chaotic behaviours is very good in the fractional-order electronic oscillator model with orders ranging from 2.8 to 3.2, and this oscillator is known to give the double-scroll-like chaotic attractors in the range $1.18 > a > 0.2$.

4. Chaotic control of the fractional-order electronic oscillator

Chaotic control[15,16] and synchronization[17−19] have attracted more and more attention from various disciplines in science and engineering since the pioneering work of Ott *et al.*[20] In this paper, we briefly discuss the issue of controlling the fractional-order chaotic electronic oscillator to its equilibrium point $(f(x_1), 0, 0)$, where

$$f(x_1) = \mathrm{sgn}(x_1) = \left\{ \begin{array}{ll} 1, & x_1 \geq 0 \\ -1, & x_1 < 0 \end{array} \right\}.$$

In recent years, adaptive Lyapunov-based methods have aroused an increasing interest due to the apparition of a systematic controller design method known as backstepping. The backstepping[9−11] is a recursive procedure for systematically selecting the control of Lyapunov function (clf) that allows the design of adaptive controllers for a class of nonlinear process. In the following simulation, we use the backstepping method to control the fractional order chaotic electronic oscillator.

Introducing a control signal u into the first equation in Eq.(3) ($\alpha = \beta = \gamma = 1$), we have

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2 + u, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_3, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= -a[x_3 + x_2 + x_1 - f(x_1)]. \end{aligned} \tag{4}$$

According to the design algorithm in Ref.[9], we can choose such a control signal u, that the derivative of the Lyapunov function is equal to zero, i.e.

$$u = (2a + 1)[f(x_1) - x_1] - x_2. \tag{5}$$

First, we show the effectiveness of the designed controller in stabilizing the chaotic trajectories of an integer order oscillator to its corresponding equilibrium point. Then, we use the same controller for the fractional-order (2.7 and 3.2) oscillator, and show its effectiveness in stabilizing the fractional-order oscillator trajectories.

In Fig.6, we show the behaviours of the output trajectories x_1 and x_3 of the chaotic oscillator for a=0.8, as the control signal is applied at t=500. Stability analysis easily shows that the equilibrium point (1, 0, 0) of the controlled integer-order oscillator is stable.

In Figs.7 and 8, we demonstrate the effectiveness of the same controller in stabilizing the chaotic output trajectories x_1 and x_2 of the fractional-order oscillator when its order becomes 2.7 and 3.2, respectively.

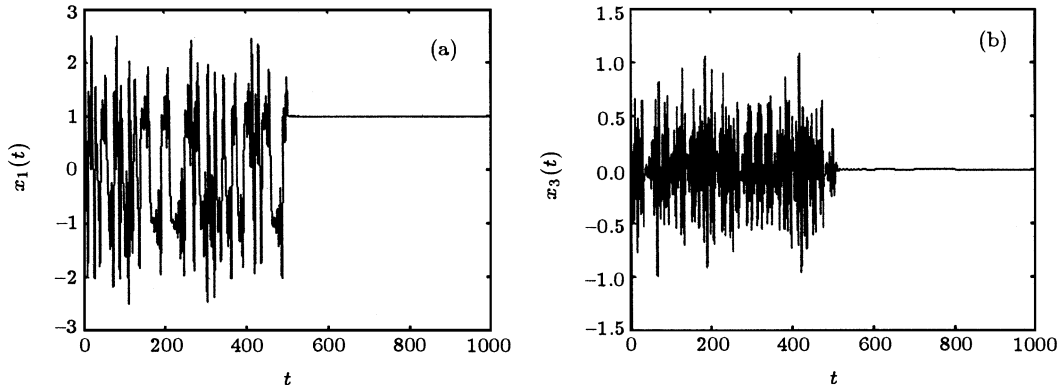


Fig.6. Chaotic output, fractional-order oscillator with an order being 3, a=0.8.

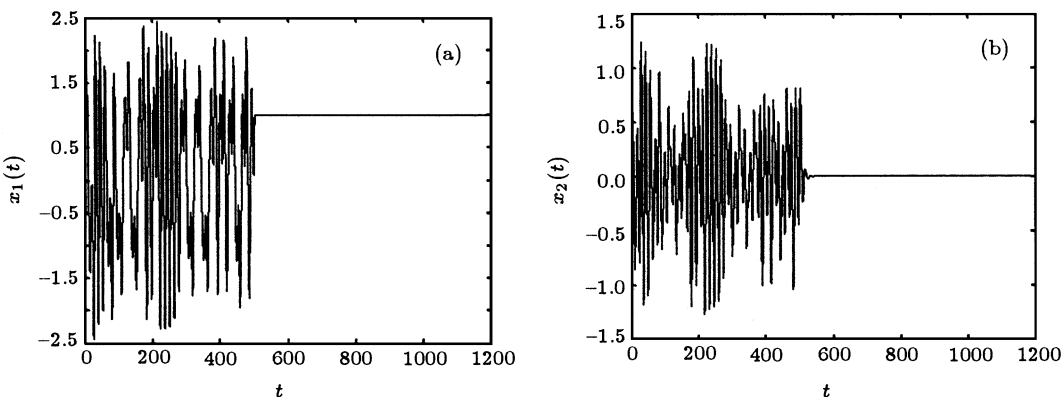


Fig.7. Chaotic output, fractional-order oscillator with an order being 2.7 ($\alpha = \beta = \gamma = 0.9$), a=0.9.

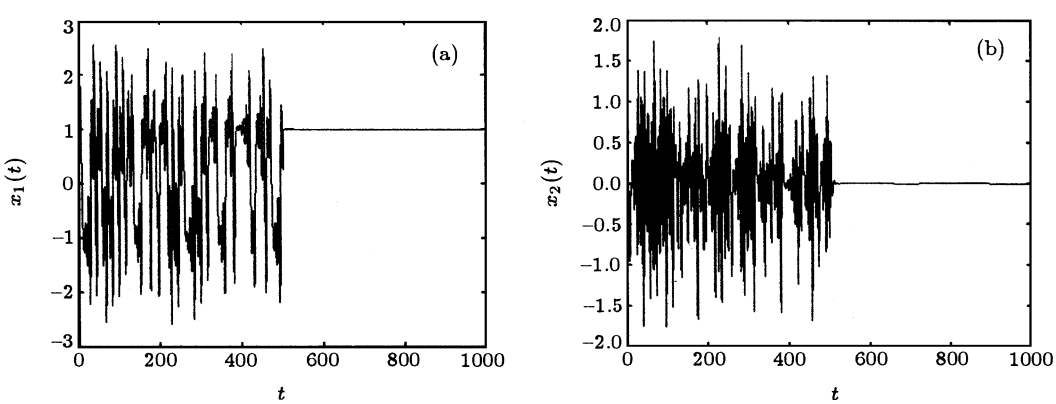


Fig.8. Chaotic output, fractional-order oscillator with an order being 3.2, a=1.18.

As we can see from Figs.7 and 8, the designed chaotic controller can effectively control the fractional order electronic oscillator to its equilibrium.

5. Conclusions

In this paper, we study the chaotic dynamics of a fractional-order electronic oscillator. We find that the chaos exists in this oscillator with an order being less than 3. In addition, we numerically simulate the continuance of the chaotic behaviours in the electronic oscillator model with orders ranging from 2.8 to 3.2. An effective recursive "backstepping" control of chaos in this fractional-order chaotic oscillator is also presented. Simulation results show that the controller designed can stabilize the fractional-order chaotic oscillator to the equilibrium. The further study of the control and synchronization of chaos in fractional-order

dynamical systems is an interesting topic in future.

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