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Discrete variational principle and the first integrals of the conservative holonomic systems in event space*

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It is shown in this paper that first integrals of discrete equation of motion for the conservative holonomic systems can be determined explicitly by investigating the invariance properties of the discrete Lagrangian in event space. The result obtained is a discrete analogue of Noether's theorem in the calculus of variations. Two examples are given to illustrate the applications of the result.

Keywords: event space, discrete mechanics, conservative holonomic system, Noether's theorem, first integral
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1. Introduction

Due to its applications to optimization and engineering problems, and because of recent emphases on numerical methods, the discrete calculus of variations becomes an important mathematical discipline[1,2] in analyses of discrete systems, the systems that are governed by difference equations. Briefly, the central problem of the discrete calculus of variations is to determine a finite sequence $q_{M-1}, q_M, \cdots, q_N$ of real numbers for which the sum

$$J\{q_k\} = \sum_{k=M}^{N} F(k, q_k, q_{k-1}) \tag{1}$$

is extremal, where the given function $F(x, y, z)$ is assumed to be continuously differentiable. A necessary condition for J to have an extremum for a given sequence $\{q_k\}(k = M-1, M, \cdots, N)$ is that $\{q_k\}$ satisfy the second-order difference equation

$$F_y(k, q_k, q_{k-1}) + F_z(k+1, q_{k+1}, q_k) = 0, \quad (k = M, M+1, \cdots, N) \tag{2}$$

where we have denoted

$$F_y(k, q_k, q_{k-1}) = \frac{\partial F(x, y, z)}{\partial y} \mid_{x=k, y=q_k, z=q_{k-1}}, \tag{3}$$

and

$$F_z(k+1, q_{k+1}, q_k) = \frac{\partial F(x, y, z)}{\partial z} \mid_{x=k+1, y=q_{k+1}, z=q_k}. \tag{4}$$

Equation (2), which is in general nonlinear, is called the discrete Euler equation because of its similarity to the classical Euler equation in the calculus of variations. Its derivation has been carried out (see Ref.[1]) by treating J as an ordinary function of the $N - M + 2$ variables $q_{M-1}, q_M, \cdots, q_N$ and applying the usual rules of the calculus, namely

$$\frac{\partial J}{\partial q_k} = 0. \quad (k = M-1, M, \cdots, N). \tag{5}$$

In 1973 Logan[3] extended Noether's theorem to a discrete case and showed that first integrals of Eq.(2) could be obtained by studying the invariance properties of $F(x, y, z)$, and further studied multi-freedom and higher-order problems. In 1981 Maeda[4] constructed the constants of motion for the discrete time system in the case of continuous symmetries. In the literature [5], we studied the first integrals of the discrete nonconservative and nonholonomic systems.

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In view of the face that only the momentum integrals can be obtained by the above approaches, in this paper we address the problem of constructing first integrals of discrete equations of motion for conservative holonomic systems by investigating the invariance properties of the discrete Lagrangian in event space. We can obtain not only the momentum integrals, but also the energy integral. In Section 2, by performing an analogue of the method and the process in continuous mechanics, a discrete Lagrange–D'Alembert principle and a discrete Euler–Lagrange equation in event space are obtained. In Section 3, by investigating the invariance properties of the discrete Lagrangian in event space, we give the discrete first integrals of systems. In Section 4, two simple examples are presented to illustrate the applications of the results. Finally, in Section 5, a summary is given.

2. The discrete variational principle of conservative holonomic systems in event space

We study a mechanical system whose configuration is determined by the n generalized coordinates q_s $(s = 1, 2, \cdots, n)$ and construct a space $\boldsymbol{R}^{n+1}$ (event space) in which the coordinates of a point are q_s and t. We introduce the notion

$$x^1 = t, \quad x^{s+1} = q_s, \quad (s = 1, 2, \cdots, n) \tag{6}$$

then all the variables x^α $(\alpha = 1, 2, \cdots, n+1)$ may be given as a function of some parameter τ. A choice of it has no special meaning, e.g. see Ref.[6]. Let

$$x^\alpha = x^\alpha(\tau), \quad (\alpha = 1, 2, \cdots, n+1) \tag{7}$$

be such curves of class C^2 such that

$$(x^\alpha)' = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}, \tag{8}$$

where the $(x^\alpha)'$ are not all zero simultaneously. We set

$$\dot{x}^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \frac{(x^\alpha)'}{(x^1)'}, \tag{9}$$

where $\dot{x^\alpha}$ denotes the derivative with respect to t and $(x^\alpha)'$ denotes the derivative with respect to τ.

It is straightforward to show that for any function $F = F(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$,[7,8] we have

$$\tilde{F}[x^\alpha, (x^\alpha)'] = (x^1)' F\left[x^\alpha, \frac{(x^2)'}{(x^1)'}, \cdots, \frac{(x^{n+1})'}{(x^1)'}\right]. \tag{10}$$

Obviously, the function $\tilde{F}$ is zero order homogenous for $(x^\alpha)'$. For the Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of a given system in configuration space, the Lagrangian $\Lambda[x^\alpha, (x^\alpha)']$ in event space are defined by the following equations:

$$\Lambda[x^\alpha, (x^\alpha)'] = (x^1)' L\left[x^\alpha, \frac{(x^2)'}{(x^1)'}, \cdots, \frac{(x^{n+1})'}{(x^1)'}\right], \tag{11}$$

and we have

$$\sum_{\alpha=1}^{n+1} \frac{\partial \Lambda}{\partial (x^\alpha)'} (x^\alpha)' = \Lambda[x^\alpha, (x^\alpha)']. \tag{12}$$

Hamilton's principle has the form

$$\int_{\tau_0}^{\tau_1} \Delta \Lambda[x^\alpha, (x^\alpha)'] \mathrm{d}\tau = 0,$$

$$\Delta x^\alpha \mid_{\tau=\tau_0} = \Delta x^\alpha \mid_{\tau=\tau_1} = 0. \tag{13}$$

The D'Alembert–Lagrange principle can be written in the form

$$\sum_{\alpha=1}^{n+1} \left[\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \Lambda}{\partial (x^\alpha)'} - \frac{\partial \Lambda}{\partial x^\alpha}\right] \Delta x^\alpha = 0, \tag{14}$$

and the parameter equations of motion for the conservative holonomic system are

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\partial \Lambda}{\partial (x^\alpha)'} - \frac{\partial \Lambda}{\partial x^\alpha} = 0. \tag{15}$$

We now define the discrete action sum in event space as

$$\tilde{J}_\mathrm{d}(x^\alpha) = \sum_{\tau=M}^{N} \Lambda_\mathrm{d}[x^\alpha(\tau - 1), x^\alpha(\tau)], \tag{16}$$

where $\Lambda_\mathrm{d}[x^\alpha(\tau-1), x^\alpha(\tau)]$ is the discrete Lagrangian in event space, which can be obtained by substituting $x^\alpha(\tau) - x^\alpha(\tau - 1)$ for $(x^\alpha)'$ and $x^\alpha(\tau - 1)$ for x^α respectively in given Lagrangian. Then the discrete Lagrange–D'Alembert principle is given by

$$\Delta \tilde{J}_\mathrm{d}(x^\alpha) = \sum_{\tau=M}^{N} \Delta \Lambda_\mathrm{d}[x^\alpha(\tau - 1), x^\alpha(\tau)] = 0. \tag{17}$$

Equation (17) is the discrete equivalent of Eq.(13).

Proposition In event space, for the conservative holonomic systems with the discrete Lagrangian $\Lambda_\mathrm{d}[x^\alpha(\tau - 1), x^\alpha(\tau)]$, the discrete Euler–Lagrange equations can be derived from the principle (17) as follows:

$$\frac{\partial \Lambda_\mathrm{d}[x^\beta(\tau - 1), x^\beta(\tau)]}{\partial x^\alpha(\tau)} + \frac{\partial \Lambda_\mathrm{d}[x^\beta(\tau), x^\beta(\tau + 1)]}{\partial x^\alpha(\tau)} = 0.$$
$$(\alpha, \beta = 1, 2, \cdots, n+1). \tag{18}$$

Proof: From Eq.(16), one has

$$\begin{aligned}&\sum_{\tau=M}^{N}\left[\frac{\partial \Lambda_{\rm d}[x^\beta(\tau-1),x^\beta(\tau)]}{\partial x^\alpha(\tau-1)}\Delta x^\alpha(\tau-1)\right.\\&\left.+\frac{\partial \Lambda_{\rm d}[x^\beta(\tau-1),x^\beta(\tau)]}{\partial x^\alpha(\tau)}\Delta x^\alpha(\tau)\right]\\=&\sum_{\tau=M}^{N-1}\left[\frac{\partial \Lambda_{\rm d}[x^\beta(\tau-1),x^\beta(\tau)]}{\partial x^\alpha(\tau)}\right.\\&\left.+\frac{\partial \Lambda_{\rm d}[x^\beta(\tau),x^\beta(\tau+1)]}{\partial x^\alpha(\tau)}\right]\Delta x^\alpha(\tau)=0,\\&(\alpha,\beta=1,2,\cdots,n+1)\end{aligned}\tag{19}$$

where we have used a discrete integration by parts (rearranging the summation) and the fact that $\Delta x^\alpha(M-1)=\Delta x^\alpha(N)=0$. If we now require that Eq.(19) be zero for any choice of $\Delta x^\alpha(\tau)$, then we obtain the discrete Euler–Lagrange equation (18) in event space.

For convenience, Eq.(18) may be rewritten as follows (In the following we will assume the summation convention of summing over repeated indices on different levels):

$$\begin{aligned}&D_2\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau)]\\&+D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau),x^\beta(\tau+1)]=0,\\&(\alpha,\beta=1,2,\cdots,n+1)\end{aligned}\tag{20}$$

where

$$D_2\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau)]=\frac{\partial \Lambda_{\rm d}[x^\beta(\tau-1),x^\beta(\tau)]}{\partial x^\alpha(\tau)},\tag{21}$$

and

$$D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau),x^\beta(\tau+1)]=\frac{\partial \Lambda_{\rm d}[x^\beta(\tau),x^\beta(\tau+1)]}{\partial x^\alpha(\tau)}.\tag{22}$$

3. First integrals of the discrete conservative holonomic systems

As in the continuous case, a systematic procedure for establishing first integrals of the discrete Euler–Lagrange equation can be developed from a direct study of the invariance properties, the discrete Lagrangian $\Lambda_{\rm d}[{\rm R}x^\alpha(\tau),x^\alpha(\tau)]$, where R is the lag operator [9] defined by

$${\rm R}^r x^\alpha(\tau)=x^\alpha(\tau-r).\quad(\alpha=1,2,\cdots,n+1).\tag{23}$$

Let us introduce the following infinitesimal transformation:

$$(x^\alpha)^*(\tau)=x^\alpha(\tau)+\varepsilon u^\alpha(\tau),\quad(\alpha=1,2,\cdots,n+1)\tag{24}$$

where ε is a parameter and $u^\alpha(\tau)=u^\alpha[x^\beta(\tau),\tau]$ is a sequence depending upon τ and $x^\beta(\tau)$, $\tau=M,\cdots,N-1$.

Definition The discrete Lagrangian $\Lambda_{\rm d}$ $[{\rm R}x^\alpha(\tau),x^\alpha(\tau)]$ is generalized-difference-invariant with respect to the infinitesimal transformation (24), if there exists such a sequence $v[x^\beta(\tau-1),x^\beta(\tau),\tau]$, $\tau=M,\cdots,N$ that

$$\begin{aligned}&\delta\Lambda_{\rm d}\{\varepsilon u^\alpha[x^\beta(\tau),\tau]\}\\=&\varepsilon\{\Delta v[x^\beta(\tau-1),x^\beta(\tau),\tau]\},\\&(\alpha,\beta=1,2,\cdots,n+1)\end{aligned}\tag{25}$$

for each τ, where $\delta\Lambda_{\rm d}$ is given by following formula:[3]

$$\begin{aligned}&\delta\Lambda_{\rm d}[\delta x^\alpha(\tau)]\\=&\{D_2\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\\&+D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau),x^\beta(\tau+1),\tau+1]\}\delta x^\alpha(\tau)\\&+\Delta\{-\delta x^\alpha(\tau-1)D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\},\\&(\alpha,\beta=1,2,\cdots,n+1)\end{aligned}\tag{26}$$

where Δ is the difference operator, $\Delta x^\alpha(\tau)=x^\alpha(\tau+1)-x^\alpha(\tau)$.

Based on the discrete Lagrange–D'Alembert principle, the following theorem concerning first integral can be proved.

Theorem If discrete Lagrangian $\Lambda_{\rm d}[Rx^\alpha(\tau),x^\alpha(\tau),\tau]$ is generalized-difference-invariant with respect to the infinitesimal transformation (24), and if Eq.(20) holds, then

$$\begin{aligned}&u^\alpha[x^\beta(\tau-1),\tau-1]D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\\&+v[x^\beta(\tau-1),x^\beta(\tau),\tau]=\text{const.}\\&(\alpha,\beta=1,2,\cdots,n+1)\end{aligned}\tag{27}$$

Proof: From Eqs.(25) and (26), one has

$$\begin{aligned}&\{D_2\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\\&+D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau),x^\beta(\tau+1),\tau+1]\}\varepsilon u^\alpha[x^\beta(\tau),\tau]\\&+\Delta\{-\varepsilon u^\alpha[x^\beta(\tau-1),\tau-1]\\&\times D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\}\\=&\varepsilon\Delta v[x^\beta(\tau-1),x^\beta(\tau),\tau].\end{aligned}$$

Simplifying and using Eq.(20), one obtains

$$\begin{aligned}&\Delta\{u^\alpha[x^\beta(\tau-1),\tau-1]D_1\Lambda_{{\rm d},\alpha}[x^\beta(\tau-1),x^\beta(\tau),\tau]\\&+v[x^\beta(\tau-1),x^\beta(\tau),\tau]\}=0,\end{aligned}\tag{28}$$

whence Eq.(27) holds. This completes the proof.

Equation (27), which is a first-order difference relation, represents a first integral of the second-order discrete Euler–Lagrange equation given by Eq.(20).

4. Illustrative examples

To illustrate relevant aspects of the theory developed in the preceding sections, two simple examples are presented.

Example 1 Consider a conservative holonomic system with its Lagrangian as follows:

$$L = \frac{1}{2}m\dot{q}^2. \tag{29}$$

Let

$$x^1 = t, \quad x^2 = q. \tag{30}$$

Hence, in event space the Lagrangian can be written as

$$\Lambda = \frac{1}{2}m\frac{[(x^2)']^2}{(x^1)'}. \tag{31}$$

Then the discrete Lagrangian is

$$\Lambda_{\rm d} = \frac{1}{2}m\frac{[x^2(\tau)-x^2(\tau-1)]^2}{x^1(\tau)-x^1(\tau-1)}. \tag{32}$$

From Eq.(20), the discrete Euler–Lagrange equations can be separated into the configuration and time components as follows:

$$m\frac{x^2(\tau)-x^2(\tau-1)}{x^1(\tau)-x^1(\tau-1)} - m\frac{x^2(\tau+1)-x^2(\tau)}{x^1(\tau+1)-x^1(\tau)} = 0, \tag{33}$$

$$\begin{aligned} &-\frac{1}{2}m\frac{[x^2(\tau)-x^2(\tau-1)]^2}{[x^1(\tau)-x^1(\tau-1)]^2} \\ &+\frac{1}{2}m\frac{[x^2(\tau+1)-x^2(\tau)]^2}{[x^1(\tau+1)-x^1(\tau)]^2} = 0. \end{aligned} \tag{34}$$

Direct verification shows that

$$\Lambda_{\rm d}[Rx^\alpha(\tau), x^\alpha(\tau), \tau] = \frac{1}{2}m\frac{[x^2(\tau)-Rx^2(\tau)]^2}{x^1(\tau)-Rx^1(\tau)} \tag{35}$$

is difference-invariant (with $v=0$) with respect to the infinitesimal transformation

$$(x^1)^*(\tau) = x^1(\tau), \tag{36}$$

$$(x^2)^*(\tau) = x^2(\tau) + \varepsilon, \tag{37}$$

and

$$(x^1)^*(\tau) = x^1(\tau) + \varepsilon, \tag{38}$$

$$(x^2)^*(\tau) = x^2(\tau), \tag{39}$$

respectively. Therefore, by use of Theorem, we immediately obtain the first integrals of Eqs.(33) and (34) respectively,

$$D_1\Lambda_{{\rm d},2}[x^\alpha(\tau-1), x^\alpha(\tau)] = \text{const}, \tag{40}$$

$$D_1\Lambda_{{\rm d},1}[x^\alpha(\tau-1), x^\alpha(\tau)] = \text{const}, \tag{41}$$

or

$$m\frac{x^2(\tau)-x^2(\tau-1)}{x^1(\tau)-x^1(\tau-1)} = \text{const}, \tag{42}$$

$$\frac{1}{2}m\frac{[x^2(\tau)-x^2(\tau-1)]^2}{[x^1(\tau)-x^1(\tau-1)]^2} = \text{const}. \tag{43}$$

Obviously, Eq.(42) is the momentum integral of the discrete system (32), and Eq.(43) is the energy integral.

Example 2 Consider the Kepler problem. Its Lagrangian is

$$L = \frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2) + \mu(q_1^2+q_2^2)^{-\frac{1}{2}}. \quad (q_1^2+q_2^2)^{\frac{1}{2}} \neq 0. \tag{44}$$

Let

$$x^1 = t, \quad x^2 = q_1, \quad x^3 = q_2. \tag{45}$$

Hence, in event space the Lagrangian can be written as follows:

$$\Lambda = \frac{1}{2}\frac{1}{(x^1)'}\{[(x^2)']^2+[(x^3)']^2\} + \mu(x^1)'[(x^2)^2+(x^3)^2]. \tag{46}$$

Then discrete Lagrangian is

$$\begin{aligned} \Lambda_{\rm d} =& \frac{1}{2}\frac{[x^2(\tau)-x^2(\tau-1)]^2+[x^3(\tau)-x^3(\tau-1)]^2}{x^1(\tau)-x^1(\tau-1)} \\ &+\mu[x^1(\tau)-x^1(\tau-1)]\{[x^2(\tau-1)]^2 \\ &+[x^3(\tau-1)]^2\}. \end{aligned} \tag{47}$$

From Eq.(20), the discrete Euler–Lagrange equations can be obtained in the following form:

$$\begin{aligned} &-\frac{1}{2}\frac{[x^2(\tau)-x^2(\tau-1)]^2+[x^3(\tau)-x^3(\tau-1)]^2}{[x^1(\tau)-x^1(\tau-1)]^2} \\ &+\frac{1}{2}\frac{[x^2(\tau+1)-x^2(\tau)]^2+[x^3(\tau+1)-x^3(\tau)]^2}{[x^1(\tau+1)-x^1(\tau)]^2} \\ &+\mu[x^2(\tau-1)]^2+\mu[x^3(\tau-1)]^2-\mu[x^2(\tau)]^2 \\ &-\mu[x^3(\tau)]^2 = 0, \end{aligned} \tag{48}$$

$$\begin{aligned} &\frac{x^2(\tau)-x^2(\tau-1)}{x^1(\tau)-x^1(\tau-1)} - \frac{x^2(\tau+1)-x^2(\tau)}{x^1(\tau+1)-x^1(\tau)} \\ &+2\mu[x^1(\tau+1)-x^1(\tau)]x^2(\tau) = 0, \end{aligned} \tag{49}$$

$$\begin{aligned} &\frac{x^3(\tau)-x^3(\tau-1)}{x^1(\tau)-x^1(\tau-1)} - \frac{x^3(\tau+1)-x^3(\tau)}{x^1(\tau+1)-x^1(\tau)} \\ &+2\mu[x^1(\tau+1)-x^1(\tau)]x^3(\tau) = 0. \end{aligned} \tag{50}$$

It is easily to verify that Eq.(47) is difference-invariant (with $v = 0$) with respect to the infinitesimal transformation

$$(x^1)^*(\tau) = x^1(\tau) + \varepsilon, \tag{51}$$
$$(x^2)^*(\tau) = x^2(\tau), \tag{52}$$
$$(x^3)^*(\tau) = x^3(\tau). \tag{53}$$

In terms of Theorem, we may immediately obtain the first integral of Eq.(48)

$$D_1 \Lambda_{\mathrm{d},1}[x^\alpha(\tau-1), x^\alpha(\tau), \tau] = \mathrm{const}, \tag{54}$$

or

$$-\frac{1}{2}\frac{[x^2(\tau)-x^2(\tau-1)]^2+[x^3(\tau)-x^3(\tau-1)]^2}{[x^1(\tau)-x^1(\tau-1)]^2} + \mu[x^2(\tau-1)]^2+\mu[x^3(\tau-1)]^2 = \mathrm{const}. \tag{55}$$

The Eq.(55) is the energy integral of the discrete system (47).

In addition, we introduce the following infinitesimal transformation:

$$(x^1)^*(\tau) = x^1(\tau), \tag{56}$$
$$(x^2)^*(\tau) = x^2(\tau) - \varepsilon x^3(\tau), \tag{57}$$
$$(x^3)^*(\tau) = x^3(\tau) + \varepsilon x^2(\tau). \tag{58}$$

After some simple calculations, one finds that Eq.(47) is also difference-invariant (with $v = 0$) with respect to the above transformation. By use of Theorem, the first integral of Eqs.(49) and (50) can be derived as follows:

$$u^2[x^\alpha(\tau-1), \tau-1] D_1 \Lambda_{\mathrm{d},2}[x^\alpha(\tau-1), x^\alpha(\tau), \tau] + u^3[x^\alpha(\tau-1), \tau-1] \times D_1 \Lambda_{\mathrm{d},3}[x^\alpha(\tau-1), x^\alpha(\tau), \tau] = \mathrm{const}, \tag{59}$$

or

$$\frac{x^2(\tau)x^3(\tau-1) - x^2(\tau-1)x^3(\tau)}{x^1(\tau) - x^1(\tau-1)} = \mathrm{const}. \tag{60}$$

Equation (60) is the angle-momentum integral of the discrete system (47).

5. Conclusion

In this paper, we define the discrete variational principle of conservative holonomic systems in event space. By investigating the invariance properties of the discrete Lagrangian, we deduce a discrete Noether theorem. This theorem may give not only the momentum integral of the equation of motion, but also the energy integral.

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