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Hojman's conservation theorems for generalized Raitzin canonical equations of motion*

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Using the Lie symmetry under infinitesimal transformations in which the time is not variable, Hojman's conservation theorems for Raitzin's canonical equations of motion in generalized classical mechanics are studied. The generalized Raitzin's canonical equations of motion are established. The determining equations of Lie symmetry under infinitesimal transformations are given. The Hojman conservation theorems of the system are established. Finally, an example is also presented to illustrate the application of the result.

Keywords: generalized classical mechanics, Hojman's conservation theorem, Raitzin's canonical equation, Lie symmetry

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1. Introduction

The generalized classical mechanical theory initiated by Ostrogradsky and Jacobi in 1859 has been developed greatly in the last 50 years, and many important results have been obtained. Especially, in physics, the Lagrangian equations with high-order derivatives in mechanics and field theories have been studied; for example, Podolsky has introduced an electromagnetic theory with second-order derivatives. And in mathematics, the description of the modern geometrical method has obtained some good results.[1−4]

The conservation law of a dynamical system plays an important role in the study of mechanics. Furthermore, this has become one of the most important directions of study in modern analytical mechanics. There are five main methods used to seek the conserved quantities of mechanical system, namely, the Noether theory,[5] the Lie method,[6] the theory of integrating factors,[7] Mei's form invariance[8] (or Mei symmetry) and the Hojman method.[9] In recent years, this kind of method has been extended to generalized classical mechanics, and significant achievements have been obtained in the study of this method.[10−20] In 1961, a Spanish scholar C M Raitzin proposed new form canonical variables r and s, and established new differential equations of motion, which are called Raitzin's canonical equations.[21] In practice, it is simpler to solve some dynamical problems by the equations than by Hamilton's canonical equations. Therefore, in analytical mechanics, Raitzin's canonical equations are of theoretical importance and applicable value. At present, some results[22−24] have been obtained. In 1990, Qiao extended Raitzin's canonical equations to the generalized classical mechanics, and obtained generalized Raitzin canonical equations.[25]

In this paper, Hojman's method is used to seek the non-Noether conserved quantity of generalized Raitzin's canonical equations.

2. Generalized Raitzin canonical equations of motion

If the configuration of a generalized mechanical system is determined by n generalized coordinates $q^j (j = 1, 2, \ldots, n)$, then the Lagrangian function is

$$L = L(t, q^j_{(0)}, q^j_{(1)}, \ldots, q^j_{(\varepsilon)}), \quad (j = 1, 2, \ldots, n), \quad (1)$$

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where

$$q^j_{(i)} = \frac{\mathrm{d}^i}{\mathrm{d}t^i} q^j, \quad (j = 1, 2, \ldots, n; i = 0, 1, \ldots, \varepsilon). \tag{2}$$

In a lower-dimensional space, the generalized non-potential force is $Q^j = Q^j(t, q^k_{(0)}, q^k_{(1)})$, while in a higher-dimensional space, the generalized non-potential force is written as

$$Q^j = Q^j(t, q^k_{(0)}, q^k_{(1)}, \ldots, q^k_{(\varepsilon)}), \quad (j, k = 1, 2, \ldots, n). \tag{3}$$

The Lagrangian equation of the generalized non-conservative system is

$$\sum_{i=0}^{\varepsilon} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\frac{\partial L}{\partial q^j_{(i)}} \right) + Q^j = 0 \quad (j = 1, 2, \ldots, n). \tag{4}$$

The generalized momentum is

$$p^{(i)}_j = \sum_{l=0}^{\varepsilon-i-1} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} \left(\frac{\partial L}{\partial q^j_{(l+i+1)}} \right),$$
$$(i = 0, 1, \ldots, \varepsilon - 1; j = 1, 2, \ldots, n). \tag{5}$$

We introduce the Raitzin canonical variables and function as follows:

$$r^{(i)}_j = \dot{p}^{(i+1)}_j = -\sum_{l=0}^{\varepsilon-i} (-1)^l \frac{\mathrm{d}^l}{\mathrm{d}t^l} \left(\frac{\partial L}{\partial q^j_{(l+i)}} \right),$$
$$s^j_{(i)} = q^j_{(i+1)}, \tag{6}$$

$$R(t, s^j_{(i)}, r^{(i)}_j) = L - r^{(i)}_j q^j_{(i)},$$
$$(i = 0, 1, \ldots, \varepsilon - 1; j = 1, 2, \ldots, n). \tag{7}$$

Hence, the generalized Ratizin canonical equations of motion are

$$s^j_{(i)} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial R}{\partial r^{(i)}_j},$$
$$r^{(i)}_j = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial R}{\partial s^j_{(i)}} - \tilde{Q}^{(i)}_j,$$
$$(i = 0, 1, \ldots, \varepsilon - 1; j = 1, 2, \ldots, n). \tag{8}$$

Suppose that the system is non-singular, and all of $s^j_{(i)}$ and $r^{(i)}_j$ will be solved from Eqs.(8) and written as

$$\dot{s}^j_{(i)} = g^j_{(i)}(t, s^j_{(i)}, r^{(i)}_j),$$
$$\dot{r}^{(i)}_j = h^{(i)}_j(t, s^j_{(i)}, r^{(i)}_j),$$
$$(i = 0, 1, \ldots, \varepsilon - 1; j = 1, 2, \ldots, n). \tag{9}$$

3. Relation between infinitesimal transformations and determining equations

Introduce the special infinitesimal transformations as follows, in which time is not variable:

$$t^* = t,$$
$$s^{*j}_{(i)}(t^*) = s^j_{(i)}(t) + \Delta s^j_{(i)},$$
$$r^{*(i)}_j(t^*) = r^{(i)}_j(t) + \Delta r^{(i)}_j. \tag{10}$$

Their expanding forms are

$$t^* = t, s^{*j}_{(i)}(t^*) = s^j_{(i)}(t) + \varepsilon \xi^j_{(i)}(t, s^j_{(i)}, r^{(i)}_j),$$
$$r^{*(i)}_j(t^*) = r^{(i)}_j(t) + \varepsilon \eta^{(i)}_j(t, s^j_{(i)}, r^{(i)}_j), \tag{11}$$

where ε is an infinitesimal parameter, and $\xi^j_{(i)}$ and $\eta^{(i)}_j$ are infinitesimal generators. Introduce an infinitesimal transformation vector of the generator

$$X^{(0)} = \xi^j_{(i)} \frac{\partial}{\partial s^j_{(i)}} + \eta^{(i)}_j \frac{\partial}{\partial r^{(i)}_j}, \tag{12}$$

and its first extension will be

$$X^{(1)} = X^{(0)} + \dot{\xi}^j_{(i)} \frac{\partial}{\partial \dot{s}^j_{(i)}} + \dot{\eta}^{(i)}_j \frac{\partial}{\partial \dot{r}^{(i)}_j}. \tag{13}$$

From the theory of invariance of differential equation under infinitesimal transformations, we know that the invariance of Eqs.(9) under the infinitesimal transformations (11) is

$$\dot{\xi}^j_{(i)} = X^{(0)}(g^j_{(i)}),$$
$$\dot{\eta}^{(i)}_j = X^{(0)}(h^{(i)}_j),$$
$$(i = 0, 1, 2, \ldots, \varepsilon - 1; j = 1, 2, \ldots, n). \tag{14}$$

Equations (14) are called the Lie symmetrical determining equations of the generalized Raitzin canonical equations (8). If the generators $\xi^j_{(i)}$ and $\eta^{(i)}_j$ of the infinitesimal transformations (11) satisfy the determining equations (14), then the transformations (11) are called Lie symmetrical transformations.

Theorem 1 For the generalized Raitzin canonical equations (8), if the generators $\xi^j_{(i)}$ and $\eta^{(i)}_j$ of the infinitesimal transformations (11) satisfy the determining equations (14), then the corresponding symmetries are Lie symmetries of Eqs.(8).

4. Hojman's conservation theorems of the generalized Raitzin canonical equation of motion

We now obtain directly the non-Noether conserved quantities from the Lie symmetries of Eqs.(8).

Theorem 2 For the generalized Raitzin canonical equations (8), if the generators $\xi^j_{(i)}$ and $\eta^{(i)}_j$ of the infinitesimal transformations (11) satisfy the determining equations (14), and there exists a function $\mu(s^j_{(i)}, r^{(i)}_j)$ satisfying the following condition:

$$\frac{1}{\mu}\frac{\partial}{\partial s^j_{(i)}}(\mu g^j_{(i)}) + \frac{1}{\mu}\frac{\partial}{\partial r^{(i)}_j}(\mu h^{(i)}_j) = 0, \tag{15}$$

then Eqs.(8) have the Lie symmetrical non-Noether conserved quantities as follows:

$$I = \frac{1}{\mu}\frac{\partial}{\partial s^j_{(i)}}(\mu\xi^j_{(i)}) + \frac{1}{\mu}\frac{\partial}{\partial r^{(i)}_j}(\mu\eta^{(i)}_j) = \text{const}. \tag{16}$$

Proof

Differentiating the formula (16) with respect to time t, we have

$$\begin{aligned}\frac{\mathrm{d}I_H}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)\xi^j_{(i)} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)\eta^{(i)}_j \\ &+ \frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\dot{\xi}^j_{(i)} \\ &+ \frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\dot{\eta}^{(i)}_j + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi^j_{(i)}}{\partial s^j_{(i)}} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta^{(i)}_j}{\partial r^{(i)}_j}. \end{aligned}\tag{17}$$

From Eqs.(9) and (12), the determining equations (14) can be written in the form

$$\begin{aligned}&\dot{\xi}^j_{(i)} = \xi^l_{(k)}\frac{\partial g^j_{(i)}}{\partial s^l_{(k)}} + \eta^{(k)}_l\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l},\\ &\dot{\eta}^{(i)}_j = \xi^l_{(k)}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}} + \eta^{(k)}_l\frac{\partial h^{(i)}_j}{\partial r^{(k)}_l},\\ &(j,l = 1,2,\ldots,n; i,k = 0,1,2,\ldots,\varepsilon-1).\end{aligned}\tag{18}$$

So we have

$$\begin{aligned}\frac{\partial\dot{\xi}^j_{(i)}}{\partial s^j_{(i)}} =& \frac{\partial\xi^l_{(k)}}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial s^l_{(k)}} + \frac{\partial\eta^{(k)}_l}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l} \\ &+ \xi^l_{(k)}\frac{\partial^2 g^j_{(i)}}{\partial s^l_{(k)}\partial s^j_{(i)}} + \eta^{(k)}_l\frac{\partial^2 g^j_{(i)}}{\partial r^{(k)}_l\partial s^j_{(i)}},\\ \frac{\partial\dot{\eta}^{(i)}_j}{\partial r^{(i)}_j} =& \frac{\partial\xi^l_{(k)}}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}} + \frac{\partial\eta^{(k)}_l}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial r^{(k)}_l} \\ &+ \xi^l_{(k)}\frac{\partial^2 h^{(i)}_j}{\partial s^l_{(k)}\partial r^{(i)}_j} + \eta^{(k)}_l\frac{\partial^2 h^{(i)}_j}{\partial r^{(k)}_l\partial r^{(i)}_j}.\end{aligned}\tag{19}$$

Noticing

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi^j_{(i)}}{\partial s^j_{(i)}} =& \frac{\partial^2\xi^j_{(i)}}{\partial s^j_{(i)}\partial s^l_{(k)}}g^l_{(k)} \\ &+ \frac{\partial^2\xi^j_{(i)}}{\partial s^j_{(i)}\partial r^{(k)}_l}h^{(k)}_l + \frac{\partial^2\xi^j_{(i)}}{\partial s^j_{(i)}\partial t},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta^{(i)}_j}{\partial r^{(i)}_j} =& \frac{\partial^2\eta^{(i)}_j}{\partial r^{(i)}_j\partial s^l_{(k)}}g^l_{(k)} \\ &+ \frac{\partial^2\eta^{(i)}_j}{\partial r^{(i)}_j\partial r^{(k)}_l}h^{(k)}_l + \frac{\partial^2\eta^{(i)}_j}{\partial r^{(i)}_j\partial t},\end{aligned}\tag{20}$$

$$\begin{aligned}\dot{\xi}^j_{(i)} &= \frac{\partial\xi^j_{(i)}}{\partial s^l_{(k)}}g^l_{(k)} + \frac{\partial\xi^j_{(i)}}{\partial r^{(k)}_l}h^{(k)}_l + \frac{\partial\xi^j_{(i)}}{\partial t},\\ \dot{\eta}^{(i)}_j &= \frac{\partial\eta^{(i)}_j}{\partial s^l_{(k)}}g^l_{(k)} + \frac{\partial\eta^{(i)}_j}{\partial r^{(k)}_l}h^{(k)}_l + \frac{\partial\eta^{(i)}_j}{\partial t},\end{aligned}\tag{21}$$

then we have

$$\begin{aligned}\frac{\partial\dot{\xi}^j_{(i)}}{\partial s^j_{(i)}} =& \frac{\partial^2\xi^j_{(i)}}{\partial s^l_{(k)}\partial s^j_{(i)}}g^l_{(k)} + \frac{\partial\xi^j_{(i)}}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial s^j_{(i)}} + \frac{\partial^2\xi^j_{(i)}}{\partial r^{(k)}_l\partial s^j_{(i)}}h^{(k)}_l \\ &+ \frac{\partial\xi^j_{(i)}}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial s^j_{(i)}} + \frac{\partial^2\xi^j_{(i)}}{\partial t\partial s^j_{(i)}} \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi^j_{(i)}}{\partial s^j_{(i)}} + \frac{\partial\xi^j_{(i)}}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial s^j_{(i)}} + \frac{\partial\xi^j_{(i)}}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial s^j_{(i)}},\\ \frac{\partial\dot{\eta}^{(i)}_j}{\partial r^{(i)}_j} =& \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta^{(i)}_j}{\partial r^{(i)}_j} + \frac{\partial\eta^{(i)}_j}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial r^{(i)}_j} + \frac{\partial\eta^{(i)}_j}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial r^{(i)}_j}.\end{aligned}\tag{22}$$

Substituting Eqs.(19) into Eqs.(22), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\xi^j_{(i)}}{\partial s^j_{(i)}} =& \frac{\partial\eta^{(k)}_l}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial r^{(k)}_j} - \frac{\partial\xi^j_{(i)}}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial s^j_{(i)}} \\ &+ \xi^l_{(k)}\frac{\partial^2 g^j_{(i)}}{\partial s^l_{(k)}\partial s^j_{(i)}} + \eta^{(k)}_l\frac{\partial^2 g^j_{(i)}}{\partial r^{(k)}_l\partial s^j_{(i)}},\\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\eta^{(i)}_j}{\partial r^{(i)}_j} =& \frac{\partial\xi^l_{(k)}}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}} - \frac{\partial\eta^{(i)}_j}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial r^{(i)}_j} \\ &+ \xi^l_{(k)}\frac{\partial^2 h^{(i)}_j}{\partial s^l_{(k)}\partial r^{(i)}_j} + \eta^{(k)}_l\frac{\partial^2 h^{(i)}_j}{\partial r^{(k)}_l\partial r^{(i)}_j}.\end{aligned}\tag{23}$$

Next, substituting Eqs.(18) and (23) into Eqs.(17), we have

$$\begin{aligned}\frac{\mathrm{d}I_H}{\mathrm{d}t}=&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)\xi^j_{(i)}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)\eta^{(i)}_j\\&+\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\left(\xi^l_{(k)}\frac{\partial g^j_{(i)}}{\partial s^l_{(k)}}+\eta^{(k)}_l\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l}\right)\\&+\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\left(\xi^l_{(k)}\frac{\partial h^{(i)}_l}{\partial s^l_{(k)}}+\eta^{(k)}_l\frac{\partial h^{(i)}_j}{\partial r^{(k)}_l}\right)\\&+\xi^l_{(k)}\frac{\partial^2 g^j_{(i)}}{\partial s^l_{(k)}\partial s^j_{(i)}}+\eta^{(k)}_l\frac{\partial^2 g^j_{(i)}}{\partial r^{(k)}_l\partial s^j_{(i)}}\\&+\xi^l_{(k)}\frac{\partial^2 h^{(i)}_j}{\partial s^l_{(k)}\partial r^{(i)}_j}+\eta^{(k)}_l\frac{\partial^2 h^{(i)}_j}{\partial r^{(k)}_l\partial r^{(i)}_j}\\=&\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)\dot{s}^l_{(k)}\xi^j_{(i)}\\&+\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)\dot{r}^{(k)}_l\xi^j_{(i)}\\&+\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)\dot{s}^l_{(k)}\eta^{(i)}_j\\&+\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)\dot{r}^{(k)}_l\eta^{(i)}_j\\&+\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\left(\xi^l_{(k)}\frac{\partial g^j_{(i)}}{\partial s^l_{(k)}}+\eta^{(k)}_l\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l}\right)\\&+\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}(\xi^l_{(k)}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}}\\&+\eta^{(k)}_l\frac{\partial h^{(i)}_j}{\partial r^{(k)}_l})+\xi^l_{(k)}\frac{\partial^2 g^j_{(i)}}{\partial s^l_{(k)}\partial s^j_{(i)}}\\&+\eta^{(k)}_l\frac{\partial^2 g^j_{(i)}}{\partial r^{(k)}_l\partial s^j_{(i)}}+\xi^l_{(k)}\frac{\partial^2 h^{(i)}_j}{\partial s^l_{(k)}\partial r^{(i)}_j}\\&+\eta^{(k)}_l\frac{\partial^2 h^{(i)}_j}{\partial r^{(k)}_l\partial r^{(i)}_j}.\end{aligned}\tag{24}$$

From the condition (15), we have

$$\begin{aligned}&\frac{\partial^2 g^j_{(i)}}{\partial s^j_{(i)}\partial s^l_{(k)}}+\frac{\partial^2 h^{(i)}_j}{\partial r^{(i)}_j\partial s^l_{(k)}}\\=&-\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)g^j_{(i)}-\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial s^l_{(k)}}\\&-\frac{1}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)h^{(i)}_j-\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}},\end{aligned}\tag{25}$$

$$\begin{aligned}&\frac{\partial^2 g^j_{(i)}}{\partial s^j_{(i)}\partial r^{(k)}_l}+\frac{\partial^2 h^{(i)}_j}{\partial r^{(i)}_j\partial r^{(k)}_l}\\=&-\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)g^j_{(i)}-\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l}\\&-\frac{1}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)h^{(i)}_j\\&-\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial r^{(k)}_l}.\end{aligned}\tag{26}$$

Noticing

$$\begin{aligned}&\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)g^l_{(k)}\xi^j_{(i)}=\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)g^j_{(i)}\xi^l_{(k)},\\&\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^j_{(i)}}\right)h^{(k)}_l\xi^j_{(i)}=\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)h^{(i)}_j\xi^l_{(k)},\\&\frac{\partial}{\partial s^l_{(k)}}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)g^l_{(k)}\eta^{(i)}_j=\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial s^{(i)}_j}\right)g^j_{(i)}\xi^{(k)}_l,\\&\frac{\partial}{\partial r^{(k)}_l}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)h^{(k)}_l\eta^{(i)}_j=\frac{\partial}{\partial r^{(k)}_j}\left(\frac{1}{\mu}\frac{\partial\mu}{\partial r^{(i)}_j}\right)h^{(i)}_j\eta^{(k)}_l,\end{aligned}\tag{27}$$

and substituting Eqs.(25), (26) and (27) into the formula (24), we obtain

$$\frac{\mathrm{d}I_H}{\mathrm{d}t}=0,\quad\text{i.e.}\quad I_H=\text{const.}\tag{28}$$

Therefore, Eqs.(8) have the Lie symmetrical non-Noether conserved quantities (16). QED.

Theorem 3 For the generalized Raitzin canonical equations (8), if the generators $\xi^j_{(i)}$ and $\eta^{(i)}_j$ of the infinitesimal transformations (11) satisfy the determining equations (14) and the following condition:

$$\frac{\partial g^j_{(i)}}{\partial s^j_{(i)}}+\frac{\partial h^{(i)}_j}{\partial r^{(i)}_j}=0,\tag{29}$$

then Eqs.(8) have the Lie symmetrical non-Noether conserved quantities as follows:

$$I'_H=\frac{\partial\xi^j_{(i)}}{\partial s^j_{(i)}}+\frac{\partial\eta^{(i)}_j}{\partial r^{(i)}_j}=\text{const.}\tag{30}$$

Proof Differentiating Eqs.(30) with respect to

time t, and make use of Eqs.(23), we have

$$\begin{aligned}\frac{\mathrm{d}I'_H}{\mathrm{d}t} =&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial g^j_{(i)}}{\partial s^j_{(i)}} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \eta^{(i)}_j}{\partial r^{(i)}_j}\\ =&\frac{\partial \eta^{(k)}_l}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l} - \frac{\partial \eta^{(i)}_j}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial r^{(i)}_j}\\ &+\frac{\partial \xi^l_{(k)}}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}} - \frac{\partial \xi^j_{(i)}}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial s^j_{(i)}}\\ &+\xi^l_{(k)}\frac{\partial}{\partial s^l_{(k)}}\left(\frac{\partial g^j_{(i)}}{\partial s^j_{(i)}} + \frac{\partial h^{(i)}_j}{\partial r^{(i)}_j}\right)\\ &+\eta^{(k)}_l\frac{\partial}{\partial r^{(k)}_l}\left(\frac{\partial g^j_{(i)}}{\partial s^j_{(i)}} + \frac{\partial h^{(i)}_j}{\partial r^{(i)}_j}\right). \end{aligned} \tag{31}$$

Noticing

$$\begin{aligned}&\frac{\partial \eta^{(k)}_l}{\partial s^j_{(i)}}\frac{\partial g^j_{(i)}}{\partial r^{(k)}_l} = \frac{\partial \eta^{(i)}_j}{\partial s^l_{(k)}}\frac{\partial g^l_{(k)}}{\partial r^{(i)}_j},\\ &\frac{\partial \xi^l_{(k)}}{\partial r^{(i)}_j}\frac{\partial h^{(i)}_j}{\partial s^l_{(k)}} = \frac{\partial \xi^j_{(i)}}{\partial r^{(k)}_l}\frac{\partial h^{(k)}_l}{\partial s^j_{(i)}}, \end{aligned} \tag{32}$$

and substituting Eqs.(29) and (32) into Eqs.(31), we obtain

$$\frac{\mathrm{d}I'_H}{\mathrm{d}t} = 0, \quad \text{i.e.} \quad I'_H = \text{const.} \tag{33}$$

QED.

5. Example

Suppose that the Lagrange function of a mechanical system is

$$L = \frac{1}{2}\dot{y}^2 + \frac{1}{2}\ddot{y}^2, \tag{34}$$

and that the generalized non-potential forces are

$$Q = -\ddot{y}. \tag{35}$$

We will find the non-Noether conserved quantities of the system.

Solution: The first step is to establish the generalized Raitzin canonical equations.

Letting

$$\begin{aligned}&q^1_{(1)} = \dot{y}, \quad q^1_{(2)} = \ddot{y},\\ &q^1_{(1)} = s^1_{(0)}, \quad q^1_{(2)} = s^1_{(1)},\\ &\tilde{Q}^{(0)}_1 = 0, \quad \tilde{Q}^{(1)}_1 = -s^1_{(1)}. \end{aligned} \tag{36}$$

Then

$$L = \frac{1}{2}(q^1_{(1)})^2 + \frac{1}{2}(q^1_{(2)})^2. \tag{37}$$

We introduce the Raitzin canonical variables and function as follows:

$$\begin{aligned}&r^{(0)}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^1_{(1)}} - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial q^1_{(2)}} = s^1_{(1)} - \dot{s}^1_{(2)},\\ &r^{(1)}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial q^1_{(2)}} = \dot{s}^1_{(1)},\\ &R = \frac{1}{2}(s^1_{(0)})^2 + \frac{1}{2}(s^1_{(1)})^2 - r^{(0)}_1 q^1_{(0)} - r^{(1)}_1 q^1_{(1)}. \end{aligned} \tag{38}$$

Then we have

$$\begin{aligned}&r^{(0)}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^1_{(0)}} - \tilde{Q}^{(0)}_1 = s^1_{(1)},\\ &r^{(1)}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial s^1_{(1)}} - \tilde{Q}^{(1)}_1 = s^1_{(2)} + s^1_{(1)},\\ &s^1_{(0)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r^{(0)}_1} = q^1_{(1)},\\ &s^1_{(1)} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial r^{(1)}_1} = q^1_{(2)}. \end{aligned} \tag{39}$$

From Eqs.(9), we obtain

$$\begin{aligned}&\dot{r}^{(0)}_1 = \dot{s}^1_{(1)} = r^{(1)}_1 = h^{(0)}_1,\\ &\dot{r}^{(1)}_1 = \dot{s}^1_{(2)} + \dot{s}^1_{(1)} = r^{(1)}_1 = h^{(1)}_1,\\ &\dot{s}^1_{(0)} = \dot{q}^1_{(1)} = s^1_{(1)} = g^1_{(0)},\\ &\dot{s}^{(1)}_1 = \dot{q}^1_{(2)} = \dot{s}^1_{(1)} = r^{(1)}_1 = g^{(1)}_1. \end{aligned} \tag{40}$$

The second step is to establish the determining equations. From Eqs.(14) we have

$$\begin{aligned}&\dot{\xi}^1_{(0)} = \xi^1_{(1)}, \quad \dot{\xi}^1_{(1)} = \eta^{(1)}_1 \quad (\dot{\xi}^1_{(1)} = \xi^1_{(2)}),\\ &\dot{\eta}^{(0)}_1 = \eta^{(1)}_1, \quad \dot{\eta}^{(1)}_1 = \eta^{(1)}_1. \end{aligned} \tag{41}$$

The solutions of Eqs.(41) are as follows:

$$\begin{aligned}&\xi^1_{(0)} = 1, \quad \xi^1_{(1)} = 0 \quad (\xi^1_{(2)} = 0),\\ &\eta^{(0)}_1 = 1, \quad \eta^{(1)}_1 = 0. \end{aligned} \tag{42}$$

The third step is to seek the non-Noether conserved quantity of the system.

The Eq.(15) gives

$$\mu = [2s^1_{(0)} - s^1_{(2)}]^2 \cdot \exp(-t). \tag{43}$$

Substituting Eqs.(42) and (43) into formula (16), we obtain

$$\begin{aligned}I_H =&\frac{1}{\mu}\frac{\partial \mu}{\partial s^1_{(0)}}\xi^1_{(0)} + \frac{1}{\mu}\frac{\partial \mu}{\partial s^1_{(2)}}\xi^1_{(2)}\\ =&4(2s^1_{(0)} - s^1_{(2)})^{-1} = \text{const}, \end{aligned} \tag{44}$$

or

$$I_H = 4(2\dot{y} - \dddot{y})^{-1} = \text{const}. \tag{45}$$

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