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The third-order Lagrange equation for mechanical systems of variable mass

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In this paper, based on the third-order D'Alembert–Lagrange principle for mechanical systems of variable mass, the third-order Lagrange equations of mechanical systems of variable mass are obtained. From the equations the motion of mechanical systems of variable mass can be studied. In addition, the equations may enrich the theory of third-order differential equation.

Keywords: mechanical system of variable mass, third-order D'Alembert–Lagrange principle, time rate of force, third-order Lagrange equations

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1. Introduction

In order to meet the needs of space detection and the development of space technology, researches on the mechanical system of variable mass have become more and more important. In analytical mechanics, the Euler–Lagrange equation, Nielsen equation and Appell equation, which contain the second-order time differentiation of generalized coordinates, have been already obtained for the mechanical system of variable mass.[1] In recent years, research works on the mechanical system of variable mass have also often been seen in periodicals. In 1999, starting from the Lagrangian equations for the mechanical systems with variable mass, the general energy change law was formulated by Musicki.[2] In 2001, the Lie symmetries and conserved quantities of relativistic rotational variable-mass system were investigated by Fang *et al.*[3] In 2002, the form invariance of Gibbs–Appell equations for variable mass holonomic system under the condition of infinitesimal transformations of groups has been studied by Li *et al.*[4] Later on, under the condition of Lie symmetry of the infinitesimal transformations in which time is not variable, the problem of non-Neother symmetrical conserved quantity for holonomic mechanical system with variable mass was discussed by Xu.[5] In 2004, the definition and criterion of the Mei symmetry of variable mass systems with unilateral holonomic constraints have been investigated by Li *et al.*[6] And the structure equation and conserved quantity of the Mei symmetry have been obtained. All of these studies show that the studies on the mechanical system of variable mass have attracted more and more interest of researchers.

In this paper, beginning with the third-order D'Alembert–Lagrange principle of mechanical system of variable mass, the third-order Lagrange's equations with differentiation with respect to generalized acceleration for the mechanical system of variable mass have been given.

2. The third-order Lagrange equations of the mechanical system of variable mass

Consider a mechanical system containing N particles, in which at time t the mass of the ith particle is $m_i (i = 1, 2, \cdots, N)$; while at time $t + \Delta t$, the mass which is separated from or incorporated into the particle is $\mathrm{d}m_i$. Then for each particle we can write the Newton's equation as follows

$$-m_i \ddot{\boldsymbol{r}}_i + \boldsymbol{F}_i + \boldsymbol{R}_i = 0, \quad (i = 1, 2, \cdots, N), \qquad (1)$$

where $\boldsymbol{F}_i$ is the active force exerted on the particle, $\boldsymbol{R}_i$ is the constraint reactive force acting on the particle.

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Differentiating Eq.(1), we have

$$-\dot{m}_i\ddot{\boldsymbol{r}}_i - m_i\dddot{\boldsymbol{r}}_i + \dot{\boldsymbol{F}}_i + \dot{\boldsymbol{R}}_i = 0, \quad (i = 1, 2, \cdots, N). \tag{2}$$

For the above equation, making the dot product with $\delta\boldsymbol{r}_i$ and summing up all equations, one has

$$\sum_{i=1}^{N}(-\dot{m}_i\ddot{\boldsymbol{r}}_i - m_i\dddot{\boldsymbol{r}}_i + \dot{\boldsymbol{F}}) \cdot \delta\boldsymbol{r}_i = 0,$$

$$(i = 1, 2, \cdots, N), \tag{3}$$

where on considering the ideal constraint condition and Jourdain ideal constraint condition,[1] we have used the result $\sum_{i=1}^{N}\dot{\boldsymbol{R}}_i \cdot \delta\boldsymbol{r}_i = 0$. Equation (3) is the third-order D'Alembert–Lagrange principle for the mechanical system of variable mass.

On the other hand, after introduction of the generalized coordinates $q_\alpha(\alpha = 1, 2, \cdots, S)$, the vector position of the ith particle can be written as

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_\alpha, t), \quad \delta\boldsymbol{r}_i = \sum_{\alpha=1}^{s}\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\delta q_\alpha,$$

$$(\alpha = 1, 2, \cdots, S). \tag{4}$$

And the corresponding velocity and acceleration are written as

$$\dot{\boldsymbol{r}}_i = \frac{\mathrm{d}\boldsymbol{r}_i}{\mathrm{d}t} = \sum_\alpha\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial\boldsymbol{r}_i}{\partial t}, \tag{5}$$

$$\delta\ddot{\boldsymbol{r}}_i = \sum_\beta\sum_\alpha\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial q_\alpha}\dot{q}_\alpha\dot{q}_\beta + \sum_\alpha\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\ddot{q}_\alpha + 2\sum_\beta\frac{\partial^2\boldsymbol{r}_i}{\partial q_\beta\partial t}\dot{q}_\beta + \frac{\partial^2\boldsymbol{r}_i}{\partial t^2}, \tag{6}$$

$$\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha} = \frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}, \quad \frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\ddot{q}_\alpha} = \frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}. \tag{7}$$

Substituting Eq.(4) into Eq.(3), we obtain

$$\sum_{i=1}^{N}\Bigg(-\dot{m}_i\sum_\alpha\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha} - m_i\sum_\alpha\dddot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha} + \dot{\boldsymbol{F}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\Bigg)\cdot\delta q_\alpha = 0. \tag{8}$$

In Eq.(8), the second term is

$$-\dddot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{\boldsymbol{r}}\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\right) + \ddot{\boldsymbol{r}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}\right) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{\boldsymbol{r}}\cdot\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\ddot{q}_\alpha}\right) + \ddot{\boldsymbol{r}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha}\right), \tag{9}$$

where we have used the results of Eq.(7). The second term on the right side of Eq.(9) can be simplified as follows:[7]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha}\right) = \frac{1}{2}\frac{\partial}{\partial\dot{q}_\alpha}(\ddot{\boldsymbol{r}}_i), \tag{10}$$

$$\ddot{\boldsymbol{r}}_i\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\dot{\boldsymbol{r}}_i}{\partial\dot{q}_\alpha} = \frac{1}{2}\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{1}{2}\ddot{\boldsymbol{r}}_i^2\right). \tag{11}$$

Substituting the results of Eqs.(9) and (11) into Eq.(8) and letting $Q_\alpha^* = \sum_i\dot{\boldsymbol{F}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}$, we find that in the holonomic system the generalized coordinates $q_1, \cdots, q_s$ are mutually independent. We can obtain the following equation from Eq.(8):

$$-\sum_i\dot{m}_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha} - \sum_i m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\ddot{q}_\alpha}\right) + \sum_i m_i\frac{1}{2}\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{1}{2}\ddot{\boldsymbol{r}}_i^2\right) + Q_\alpha^* = 0, \tag{12}$$

or

$$\sum_i m_i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{\partial\ddot{\boldsymbol{r}}_i^2}{\partial\ddot{q}_\alpha}\right) - \sum_i m_i\frac{1}{2}\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{1}{2}\ddot{\boldsymbol{r}}_i^2\right) = Q_\alpha^* - \sum_i\dot{m}_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}. \tag{13}$$

Under the general circumstance, the mass of the particle can be a function of generalized coordinates, generalized velocities and time, i.e.

$$m_i = m_i(q_\alpha, \dot{q}_\alpha, t), \tag{14}$$

$$\dot{m}_\alpha = \sum_\alpha\frac{\partial m_i}{\partial q_\alpha}\dot{q}_\alpha + \sum_\alpha\frac{\partial m_i}{\partial\dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial m_i}{\partial t}. \tag{15}$$

Now introduce a new denotation of differentiation. π is the denotation of particle derivation which considers that the mass is constant. $\frac{\mathrm{D}}{\mathrm{D}t}$ is the differentiation with respect to time, which considers that the mass is constant.[1] We can rewrite Eq.(13) as follows:

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\pi S}{\pi\ddot{q}_\alpha}\right) - \frac{1}{2}\frac{\pi S}{\pi\dot{q}_\alpha} = Q_\alpha^* - \sum_i\dot{m}_i\ddot{\boldsymbol{r}}_i\cdot\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\ddot{q}_\alpha},$$

$$(\alpha = 1, 2, \cdots, S), \tag{16}$$

where $S = \frac{1}{2}\sum_i m_i\ddot{\boldsymbol{r}}_i^2$ is the acceleration energy. Substituting Eq.(15) into Eq.(16), one has

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\pi S}{\pi\ddot{q}_\alpha}\right) - \frac{1}{2}\frac{\pi S}{\pi\dot{q}_\alpha} = Q_\alpha^* - \sum_i\left[\frac{\partial\ddot{\boldsymbol{r}}_i}{\partial\ddot{q}_\alpha}\left(\sum_\beta\frac{\partial m_i}{\partial q_\beta}\dot{q}_\beta + \sum_\beta\frac{\partial m_i}{\partial\dot{q}_\beta}\ddot{q}_\beta + \frac{\partial m_i}{\partial t}\right)\right],$$

$$(\alpha, \beta = 1, 2, \cdots, S). \tag{17}$$

Equation (17) is the third-order Lagrange equation for the mechanical system of variable mass. If we know

the time rate of force $\dot{\boldsymbol{F}}_i$, then we can use Eq.(17) to obtain the dynamical equation of the mechanical system of variable mass.

3. Discussion

Based on the third-order D'Alembert–Lagrange principle for the mechanical system of variable mass, we have obtained the third-order Lagrange equations of the mechanical system of variable mass. The equations not only enable us to obtain the equations of motion for the mechanical system of variable mass which contains the time rate of force $\dot{\boldsymbol{F}}_i$, but also enrich the theory of the third-order differential equation which contains the time rate of change of the force $\dot{\boldsymbol{F}}_i$ in analytical mechanics.

In Eq.(17), when m_i is a constant, we have $\dfrac{\partial m_i}{\partial t} = \dfrac{\partial m_i}{\partial q_\alpha} = \dfrac{\partial m_i}{\partial \dot{q}_\alpha} = 0$, $\dfrac{\mathrm{D}}{\mathrm{D}t} = \dfrac{\mathrm{d}}{\mathrm{d}t}$, $\dfrac{\pi}{\pi \ddot{q}_\alpha} = \dfrac{\partial}{\partial \ddot{q}_\alpha}$, $\dfrac{\pi}{\pi \dot{q}_\alpha} = \dfrac{\partial}{\partial \dot{q}_\alpha}$. The equations will return to the third-order Lagrange equations which is the same as the result in Ref.[7]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right) - \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha} = Q^*_\alpha,$$
$$(\alpha = 1, 2, \cdots, S). \tag{18}$$

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