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Homotopy perturbation method of equatorial eastern Pacific for the El Niño–Southern Oscillation mechanism*

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The EI Niño/La Niña and the Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. In this paper, the aim is to create an asymptotic solving method of nonlinear equation for the ENSO models. And based on a class of oscillator of the ENSO models, employing the method of homotopic mapping, the approximation solution of corresponding problem is studied. It is proven from the results that the homotopic method can be used for analysing the sea surface temperature anomaly in the equatorial eastern Pacific and the thermocline depth anomaly of the atmosphere–ocean oscillation for ENSO model.

Keywords: nonlinear, homotopic mapping, perturbation theory, El Niño–Southern Oscillation model
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1. Introduction

El Niño and Southern Oscillation (ENSO) are abnormal events occurring in the atmosphere and tropical Pacific ocean. The phenomenon of the ENSO is a very attractive object in international academic circles.[1−14] Lin and Mo *et al* considered a class of problems in atmospheric physics, oceanographical phenomena, and so on.[15−22] The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. And a positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific is considered. This anomaly reduces the zonal SST gradient and the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation changes and then reinforce the SST anomaly.

Recently, many scholars have investigated the nonlinear perturbed problem.[23] Approximation methods have been developed and refined.[24−29] Using the perturbed method, Mo *et al* considered a class of perturbed problems for atmospheric physics.[30−34] In this paper we study a class of ENSO model by using a simple and special method in perturbation theory.

2. The atmosphere–ocean oscillator of ENSO model

In the ENSO oscillator model the variations of both the eastern and western Pacific anomaly patterns should be considered. Due to the complexity of ENSO, we only study the oscillator model which overlooks the ENSO western Pacific anomaly pattern and does not consider the effect of the western Pacific in ENSO.

It is assumed that winds in the western Pacific do not affect the SST anomalies in the eastern Pacific of the region 14°W–85°W, 10°S–10°N. Thus the couple system governing the phenomena is given by[4]

$$\frac{\mathrm{d}T_{\mathrm{a}}}{\mathrm{d}t} = a_{11}T_{\mathrm{a}} + a_{12}T_{\mathrm{s}} + b_{11}T_{\mathrm{a}}T_{\mathrm{s}} + b_{12}T_{\mathrm{s}}^2, \quad (1)$$

$$\frac{\mathrm{d}T_{\mathrm{s}}}{\mathrm{d}t} = a_{21}T_{\mathrm{a}} + a_{22}T_{\mathrm{s}} + b_{21}T_{\mathrm{a}}T_{\mathrm{s}} + b_{22}T_{\mathrm{s}}^2, \quad (2)$$

where T_{a} is the local averaged temperature anomaly and T_{s} is the local averaged temperature anomaly in the equatorial eastern Pacific, the coefficients a_{ij} and b_{ij} are real non-negative (for the values and their physical sense see Ref.[4]).

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Introduce a set of homotopic mappings $H_i(T_a, T_s, p) : X \times Y \times I \to R$, $i = 1, 2$:

$$\begin{aligned} H_1(T_a, T_s, p) =& L_1(T_a, T_s) - L_1(u_0, v_0), \\ &+ p(L_1(u_0, v_0) - b_{11}T_aT_s - b_{12}T_s^2), \quad (3) \\ H_2(T_a, T_s, p) =& L_2(T_a, T_s) - L_2(u_0, v_0) \\ &+ p(L_2(u_0, v_0) - b_{21}T_aT_s - b_{22}T_s^2), \quad (4) \end{aligned}$$

where $X = [0, \infty)$, $Y = [0, \infty)$, $I = [0, 1]$, $R = (-\infty, +\infty)$, while the linear operators L_i, $i = 1, 2$, are

$$L_1(T_a, T_s) = \frac{dT_a}{dt} - a_{11}T_a - a_{12}T_s, \quad (5)$$

$$L_2(T_a, T_s) = \frac{dT_s}{dt} - a_{21}T_a - a_{22}T_s, \quad (6)$$

and (u_0, v_0) is an initial approximation of solution for the original system (1)-(2).

3. The main result

From Eqs.(5) and (6), letting

$$L_1(u_0, v_0) = 0, \quad (7)$$

$$L_2(u_0, v_0) = 0, \quad (8)$$

we have

$$u_0(t) = D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t), \quad (9)$$

$$v_0(t) = C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t), \quad (10)$$

where

$$\begin{aligned} \lambda_i =& \frac{1}{2}[(a_{1q} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4(a_{12}a_{21})}, \\ & i = 1, 2, \quad (11) \\ D_{10} =& \frac{a_{22}C_{10}}{\lambda_1 - a_{11}}, \quad D_{20} = \frac{a_{22}C_{20}}{\lambda_2 - a_{11}}, \quad (12) \end{aligned}$$

and C_{10} and C_{20} are arbitrary constants.

Obviously, from Eqs.(3) and (4), $H_i(T_a, T_s, 1) = 0$, $i = 1, 2$, are the same as the system (1)-(2). Then the solution $(T_a(t), T_s(t))$ of the system (1)-(2) is the same as the solution of $H_i(T_a, T_s, p) = 0$, $i = 1, 2$, as $p \to 1$.

Let

$$T_a = \sum_{k=0}^{\infty} (T_a(t))_k p^k, \quad (13)$$

$$T_s = \sum_{k=0}^{\infty} (T_s(t))_k p^k. \quad (14)$$

Consider Eqs.(13) and (14), then from Eqs.(3) and (4) we can develop the power series of p^1, and compare the coefficients of the same power in p^1 on the two sides of the equation obtained.

From the coefficients of zeroth-order in p, we have

$$L_i((T_a)_0, (T_s)_0) = L_i(u_0, v_0), \quad i = 1, 2.$$

Obviously, we have

$$(T_a(t))_0 = D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t), \quad (15)$$

$$(T_s(t))_0 = C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t), \quad (16)$$

where λ_i, D_{i0}, $i = 1, 2$, denote Eqs.(11) and (12); and C_{10}, C_{20} are arbitrary constants.

From Eqs.(3), (4) and (15), (16), considering the coefficients of the first-order terms in p, we obtain

$$\begin{aligned} &\frac{d(T_a)_1}{dt} - a_{11}(T_a)_1 - a_{12}(T_s)_1 \\ =& b_{11}(C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t)) \\ &\times (D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t)) \\ &+ b_{12}(C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t))^2, \quad (17) \\ &\frac{d(T_s)_1}{dt} - a_{21}(T_s)_1 - a_{22}(T_s)_1 \\ =& b_{21}(C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t)) \\ &\times (D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t)) \\ &+ b_{22}(C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t))^2. \quad (18) \end{aligned}$$

From Eqs.(17)–(18), we have

$$\begin{aligned} (T_a(t))_1 =& D_{11}\exp(\lambda_1 t) + D_{21}\exp(\lambda_2 t) \\ &+ r_{11}\exp(2\lambda_1 t) + r_{12}\exp((\lambda_1 + \lambda_2)t) \\ &+ r_{13}\exp(2\lambda_2 t), \quad (19) \\ (T_s(t))_1 =& C_{11}\exp(\lambda_1 t) + C_{21}\exp(\lambda_2 t) \\ &+ s_{11}\exp(2\lambda_1 t) + s_{12}\exp((\lambda_1 + \lambda_2)t) \\ &+ s_{13}\exp(2\lambda_2 t), \quad (20) \end{aligned}$$

where

$$r_{11}(t) = \frac{((2\lambda_1 - a_{22})b_{11} + a_{12}b_{21})C_{10}D_{10} + ((2\lambda_1 - a_{22})b_{12} + a_{12}b_{22})C_{10}^2}{(2\lambda_1 - a_{11})(2\lambda_1 - a_{22}) - a_{12}a_{21}},$$

$$s_{11}(t) = \frac{((2\lambda_1 - a_{11})b_{21} + a_{21}b_{11})C_{10}D_{10} + ((2\lambda_1 - a_{11})b_{21} + a_{21}b_{11})C_{10}^2}{(2\lambda_1 - a_{11})(2\lambda_1 - a_{22}) - a_{12}a_{21}},$$

$$
\begin{aligned}
r_{12}(t) =& \frac{((\lambda_1+\lambda_2-a_{22})b_{11}+a_{12}b_{21})(C_{10}D_{20}+C_{20}D_{10})}{(\lambda_1+\lambda_2-a_{11})(\lambda_1+\lambda_2-a_{22})-a_{12}a_{21}} \\
&+\frac{2((\lambda_1+\lambda_2-a_{22})b_{12}+a_{12}b_{22})C_{10}C_{20}}{(\lambda_1+\lambda_2-a_{11})(\lambda_1+\lambda_2-a_{22})-a_{12}a_{21}}, \\
s_{12}(t) =& \frac{((\lambda_1+\lambda_2-a_{11})b_{21}+a_{21}b_{11})(C_{10}D_{20}+C_{20}D_{10})}{(\lambda_1+\lambda_2-a_{11})(\lambda_1+\lambda_2-a_{22})-a_{12}a_{21}} \\
&+\frac{2((\lambda_1+\lambda_2-a_{11})b_{22}+a_{21}b_{12})C_{10}C_{20}}{(\lambda_1+\lambda_2-a_{11})(\lambda_1+\lambda_2-a_{22})-a_{12}a_{21}}, \\
r_{13}(t) =& \frac{((2\lambda_2-a_{22})b_{11}+a_{12}b_{21})C_{20}D_{20}+((2\lambda_2-a_{22})b_{12}+a_{12}b_{22})C_{20}^2}{(2\lambda_2-a_{11})(2\lambda_2-a_{22})-a_{12}a_{21}}, \\
s_{13}(t) =& \frac{((2\lambda_2-a_{11})b_{21}+a_{21}b_{11})C_{20}D_{20}+((2\lambda_2-a_{11})b_{22}+a_{21}b_{22})C_{20}^2}{(2\lambda_2-a_{11})(2\lambda_2-a_{22})-a_{12}a_{21}}, \\
D_{11} =& \frac{a_{22}C_{10}}{\lambda_1-a_{11}}, \quad D_{21}=\frac{a_{22}C_{20}}{\lambda_2-a_{11}},
\end{aligned}
$$

and λ_i, D_{i0}, $i=1,2$, are determined by Eqs.(11) and (12); and C_{1i}, C_{2i}, $i=0,1$, are arbitrary constants.

Thus we can obtain the first-order approximate solution for the system (1)-(2)

$$(T_{\rm a})_{\rm app}(t)=(T_{\rm a}(t))_0+(T_{\rm a}(t))_1, \tag{21}$$

$$(T_{\rm s})_{\rm app}(t)=(T_{\rm s}(t))_0+(T_{\rm s}(t))_1, \tag{22}$$

where $(T_{\rm a})_i$, $(T_{\rm s})_i$, $i=0,1$ are determined by Eqs.(15), (16), (19) and (20).

Using the same method, we can obtain higher-order approximate solutions.

4. Estimation of precision

Now we give a rough estimate of the precision of the above result.

We consider that for $\forall\varepsilon>0$, there is a positive integer N large enough, and we can obtain the Nth approximations $(T_{\rm a}^N)_{\rm app}$ and $(T_{\rm s}^N)_{\rm app}$ by using the above homotopic mapping method. Then we construct the following auxiliary functions $\alpha_i(t)$, $\beta_i(t)$, $i=1,2$:

$$\alpha_1(t)=(T_{\rm a}^N)_{\rm app}(t)-\delta\,\varepsilon, \tag{23}$$
$$\alpha_2(t)=(T_{\rm s}^N)_{\rm app}(t)-\delta\,\varepsilon, \tag{24}$$
$$\beta_1(t)=(T_{\rm a}^N)_{\rm app}(t)+\delta\,\varepsilon, \tag{25}$$
$$\beta_2(t)=(T_{\rm s}^N)_{\rm app}(t)+\delta\,\varepsilon, \tag{26}$$

where δ is a positive large enough constant, which will be determined below.

It is easy to prove that for a fixed ε, we can select a large enough δ such that

$$\frac{{\rm d}\alpha_1}{{\rm d}t}-a_{11}\alpha_1-a_{12}\alpha_1-b_{11}\alpha_1\alpha_2-b_{12}\alpha_1^2<0, \tag{27}$$
$$\frac{{\rm d}\alpha_2}{{\rm d}t}-a_{21}\alpha_1-a_{22}\alpha_2-b_{21}\alpha_1\alpha_2-b_{22}\alpha_2^2<0, \tag{28}$$
$$\frac{{\rm d}\beta_1}{{\rm d}t}-a_{11}\beta_1-a_{12}\beta_2-b_{11}\beta_1\beta_2-b_{12}\beta_1^2>0, \tag{29}$$
$$\frac{{\rm d}\beta_2}{{\rm d}t}-a_{21}\beta_1-a_{22}\beta_2-b_{21}\beta_1\beta_2-b_{22}\beta_2^2>0. \tag{30}$$

From Eqs.(27)–(30) and the theory of differential inequalities,[35] α_i and β_i, $i=1,2$ are lower and super solutions for ENSO model (1)-(2) respectively. Hence there exists a series of solution $(T_{\rm a}(t), T_{\rm s}(t))$ for Eqs.(1) and (2), which satisfy the inequalities:

$$\alpha_1(t)\le T_{\rm a}(t)\le\beta_1(t), \quad \alpha_2(t)\le T_{\rm s}(t)\le\beta_2(t). \tag{31}$$

Thus from Eqs.(31) and (23)–(26), we obtain

$$\left|T_{\rm a}(t)-(T_{\rm a}^N)_{\rm app}\right|<\varepsilon, \tag{32}$$
$$\left|T_{\rm s}(t)-(T_{\rm s}^N)_{\rm app}\right|<\varepsilon. \tag{33}$$

From Eqs.(32) and (33), we know that the approximations $(T_{\rm a}^N)_{\rm app}$, $(T_{\rm s}^N)_{\rm app}$ obtained using the homotopic method possess a higher degree of approximation.

5. Summary and discussion

It is not difficult to see that:

(i) For $a_{11}a_{22}<a_{12}a_{22}$, we can decide that the two characteristic roots λ_i, $i=1,2$ from Eqs.(12) and (13) are real but with different signs, therefore from the theory of singular point, the zero point in the phase plane of the linear system (6), (7) is an unstable saddle point.

(ii) For $a_{11}a_{22}>a_{12}a_{22}$, the two characteristic roots have different positive values λ_i, $i=1,2$, thus there is an unstable node or focus.

(iii) For $a_{11}a_{22} = a_{12}a_{22}$, there are a positive characteristic root and a zero root. Thus there exists a singular line in the phase plane and this line is unstable.

The trajectories of the phase plane for the system (7), (8) run away from the zero point.

Because the nonlinear terms of the coupled system (1)-(2) are $R_1 = (b_{11}T_aT_s + b_{12}T_s^2)$ and $R_2 = (b_{21}T_aT_s + b_{22}T_s^2)$, we have

$$\lim_{\|y\|\to 0} \frac{\|R\|}{\|y\|} = 0,$$

where

$$\|R\| = (R_1^2 + R_2^2)^{1/2}, \quad \|y\| = (T_a^2 + T_s^2)^{1/2}.$$

Therefore, the same stability behaviour occurs for the solution of the nonlinearly coupled system (1)-(2) and the linear system (7), (8). According to the above analysis, the solution of the system (1)-(2) is unstable.

From the above conclusion, we know that the sea–air oscillator in the ENSO model (1), (2), the air temperature T_a anomaly and sea surface temperature T_s anomaly in the equatorial eastern Pacific tend to be unstable and are an asymptotic unstable annual phenomenon.

We shall consider further that the qualitative and quantitative behaviours of the T_a and T_s anomalies in the equatorial eastern Pacific. The study of them will be described in another paper.

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