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# A class of integrable expanding model for the coupled AKNS–Kaup–Newell soliton hierarchy*

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An isospectral problem is established by means of a sub-algebra of loop Lie algebra $\tilde{A}_1$, from which the coupled AKNS–Kaup–Newell soliton hierarchy is derived. Subsequently, the integrable expanding model, i.e. integrable coupling, is constructed through enlarging the corresponding loop algebra into the loop Lie algebra $\tilde{A}_2$.

**Keywords:** coupled AKNS–Kaup–Newell soliton hierarchy, loop Lie algebra, integrable coupling
**PACC:** 0220, 0290

## 1. Introduction

Soliton theory is a powerful tool in explaining and describing the nonlinear phenomena in the fields of nonlinear optics, superconductivity, plasma physics, magnetic fluid, and so on, and has been studied for about 20 years. Searching for new integrable systems has been an interesting and important event during the past period. The subject of the integrable expanding model, also called integrable coupling, has been a new and significant direction in soliton theory, which originated from investigation of the centreless Virasoro symmetry algebras,[1] and can be used for obtaining some new integrable hierarchies in the Lax sense.

Letting

$$u_t = k(u) \tag{1}$$

be a given integrable system, the new bigger system

$$\begin{cases} u_t = k(u), \\ v_t = s(u, v) \end{cases} \tag{2}$$

can be called integrable coupling of the hierarchy (1), if it is still an integrable system in the Lax sense. In addition, there exists a condition for Eq.(2) to satisfy $\dfrac{\partial s}{\partial [u]} \neq 0$, which makes the non-triviality of a new system guaranteed. Here, $s(u, v)$ consists of all derivatives of $s$ with respect to $u$ or $v$, and, $[u]$ denotes a vector involving all derivatives of $u$ with respect to the space variable.

Some systematic methods in constructing the integrable coupling have been developed: for instance, the method in which the linear equations are added to the original system, the perturbation technique[1–4] and enlarging the corresponding loop Lie algebra means,[5–10] and so on. It is especially worthwhile to point out that Guo and Zhang have done a great deal of work in investigating the integrable couplings of some classical integrable systems, such as KdV, TC, TA, AKNS, KN, BPT and TB hierarchies,[5–7,9,10] etc.

The coupled AKNS–Kaup–Newell (CAKN) soliton hierarchy was first introduced by Zhang in Ref.[11], through which the so-called mixed nonlinear Schrödinger equation

$$r_t + \mathrm{i}r_{xx} + \beta(r^2 r^*)_x - 2\mathrm{i}\alpha r^2 r^* = 0$$

was obtained, and further investigation of the real form of the above system has been carried out by Ma and Zhou.[12] As is well known, there are many coupled hierarchies such as the coupled KdV hierarchy, the coupled Burgers hierarchy, the coupled Harry–Dym hierarchy, etc. Different from the former cases, the CAKN system is a new one in view of the type of the original soliton equations.

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The present paper is devoted to introducing the loop algebra $\tilde{A}_1$, from which the real form of the CAKN hierarchy is derived in Section 2. Subsequently, the integrable coupling for the CAKN hierarchy is constructed through enlarging the corresponding loop Lie algebra in Section 3. Some concluding remarks are given in Section 4.

## 2. The derivation of the CAKN system

The loop algebra $G$ is taken to be $\tilde{A}_1$, its bases are

$$h(n) = \lambda^n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$e(n) = \lambda^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$f(n) = \lambda^n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3}$$

which have the following relationships:

$$\begin{cases} [h(m), e(n)] = 2e(m+n), \\ [h(m), f(n)] = -2f(m+n), \\ [e(m), f(n)] = h(m+n). \end{cases}$$

And,

$$\deg h(n) = \deg e(n) = \deg f(n) = n.$$

Now, we consider an isospectral problem

$$\phi_x = U\phi,$$

where

$$U = \begin{pmatrix} \lambda & r \\ (\alpha\lambda+\beta)s & -\lambda \end{pmatrix} = h(1) + re(0) + \alpha f(1) + \beta s f(0), \tag{4}$$

and an auxiliary spectral problem

$$\phi_t = V\phi,$$
$$V = \begin{pmatrix} a & b \\ (\alpha\lambda+\beta)c & -a \end{pmatrix} = \sum_{n\geq 0} (a_n h(-n) + b_n e(-n) + c_n(\alpha f(-n+1) + \beta f(-n)), \tag{5}$$

in which $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$, $\lambda_t = 0$, $\alpha$, $\beta$ are constants with $\alpha^2 + \beta^2 \neq 0$.

Solving the stationary zero-curvature equation

$$V_x = [U, V]$$

gives the following recursion equations:

$$\begin{cases} b_{n+1} = \frac{1}{2} b_{nx} + r a_n, \\ c_{n+1} = -\frac{1}{2} c_{nx} + s a_n, \\ a_{nx} = \alpha(r c_n - s b_n) - \frac{1}{2}\beta(r c_{nx} + s b_{nx}). \end{cases} \tag{6}$$

Letting $a_0 = 1$, $b_0 = 0 = c_0$ be the initial data, the first few quantities are given by

$$a_1 = -\frac{1}{2}\beta rs, \quad b_1 = r, \quad c_1 = s,$$
$$a_2 = -\frac{1}{2}\alpha rs + \frac{1}{4}\beta(r s_x - r_x s) + \frac{3}{8}\beta^2 (rs)^2,$$
$$b_2 = -\frac{1}{2} r_x - \frac{1}{2}\beta r^2 s, c_2 = -\frac{1}{2} s_x - \frac{1}{2}\beta r s^2,$$
$$\cdots$$

Noting

$$(\lambda^n V)_+ = \sum_{i=0}^{n} (a_i h(n-i) + b_i e(n-i) + (\alpha\lambda+\beta) c_i f(n-i)),$$

and

$$V^{(n)} = (\lambda^n V)_+ + (-a_n + \partial^{-1}(r c_n - s b_n)) h(0),$$

according to the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0,$$

we have the following Lax integrable system:

$$u_t = \begin{pmatrix} r \\ s \end{pmatrix}_t = \begin{pmatrix} 2b_{n+1} + 2\beta r\partial^{-1}(sb_{n+1} - rc_{n+1}) \\ -2c_{n+1} + 2\beta s\partial^{-1}(sb_{n+1} - rc_{n+1}) \end{pmatrix}$$
$$= J\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = JL\begin{pmatrix} c_n \\ b_n \end{pmatrix} = JL^n\begin{pmatrix} c_1 \\ b_1 \end{pmatrix} = JL^n\begin{pmatrix} s \\ r \end{pmatrix}, \tag{7}$$

where

$$J = \begin{pmatrix} -2\beta r\partial^{-1}r & 2 + 2\beta r\partial^{-1}s \\ -2 - 2\beta s\partial^{-1}r & 2\beta s\partial^{-1}s \end{pmatrix},$$

which is a Hamiltonian operator,[13] and

$$L = \begin{pmatrix} -\frac{1}{2}\alpha\partial + \alpha s\partial^{-1}r - \frac{1}{2}\beta s\partial^{-1}r\partial & -\alpha s\partial^{-1}s - \frac{1}{2}\beta s\partial^{-1}r\partial \\ \alpha r\partial^{-1}r - \frac{1}{2}\beta r\partial^{-1}r & -\frac{1}{2}\alpha\partial - \alpha r\partial^{-1}s - \frac{1}{2}\beta r\partial^{-1}s\partial \end{pmatrix}.$$

A direct computation leads to

$$JL = \begin{pmatrix} 2\alpha r\partial^{-1}r & \partial - 2\alpha r\partial^{-1}s \\ \partial - 2\alpha s\partial^{-1}r & 2\alpha s\partial^{-1}s \end{pmatrix} = -(JL)^* = L^*J,$$

from which we can see that Eq.(7) has Liouville integrability.

## 3. The integrable coupling of the resulting system (7)

Given a loop Lie algebra $\tilde{G}$ and its sub-algebra $\tilde{G}_1$, $\tilde{G}_2$ satisfying

$$\tilde{G} \cong \tilde{A}_1, \ [\tilde{G}_1, \tilde{G}_2] \subset \tilde{G}_2, \tag{8}$$

we establish a new sub-algebra of loop algebra $\tilde{A}_2$ with the basis elements being

$$e_1(n) = \lambda^n \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_2(n) = \lambda^n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_3(n) = \lambda^n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[e_1(m), e_2(n)] = 2e_2(m+n),$$
$$[e_1(m), e_3(n)] = -2e_3(m+n),$$
$$[e_2(m), e_3(n)] = e_1(m+n).$$

Now, we construct a new sub-algebra of $\tilde{A}_2$ through expanding the algebra span$\{e_1,\ e_2,\ e_3\}$, reading as span$\{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$, in which the commutation relations are defined by

$$[e_1(m), e_2(n)] = 2e_2(m+n),$$
$$[e_1(m), e_3(n)] = -2e_3(m+n),$$
$$[e_2(m), e_3(n)] = e_1(m+n),$$
$$[e_1(m), e_4(n)] = e_5(m+n),$$
$$[e_1(m), e_5(n)] = -e_5(m+n),$$
$$[e_2(m), e_5(n)] = e_5(m+n),$$
$$[e_3(m), e_4(n)] = e_5(m+n),$$
$$[e_2(m), e_4(n)] = 0,$$
$$[e_3(m), e_5(n)] = [e_4(m), e_5(n)] = 0,$$

and $\deg e_i(n) = n$, $1 \le n \le 5$, where span represents the subspace of expansion. It is easy to see that the above relations are in accord with the relations in Eq.(3). Furthermore, there exists an equation, i.e. Jacobi identity, like

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0,$$

if we note that

$$a = \sum_{i=1}^{5} a_i e_i, \ b = \sum_{i=1}^{5} b_i e_i, \ c = \sum_{i=1}^{5} c_i e_i, \ \ 1 \le i \le 5,$$

in which $a_i$, $b_i$, $c_i$ are all arbitrary functions.

Therefore, $\tilde{G}_1 = \text{span}\ \{e_1, e_2, e_3\}$, $\tilde{G}_2 = \text{span}\ \{e_4, e_5\}$ and $\tilde{G} = \text{span}\ \{e_1, e_2, e_3, e_4, e_5\}$ satisfy the condition (8), and $\tilde{G}$ is a Lie algebra.

Accordingly, we can establish a new isospectral problem based on sub-algebra $\tilde{G}$. The resulting hierarchy of equations is the integrable coupling of the system (7).

Now we investigate the linear isospectral problem

$$\phi_x = [U, \phi], \quad \lambda_t = 0, \quad \phi_t = [V, \phi], \tag{9}$$

in which $\phi = \sum_{i=1}^{5} \phi_i e_i$, $\phi_i$ $(1 \le i \le 5)$ are arbitrary functions, $U = U(u, \lambda) \in \tilde{G}$, $V = V(u, \lambda) \in \tilde{G}$, and $u = (u_1, \ldots, u_p)^{\mathrm{T}}$ is a functional vector, $\lambda$ is isospectral parameter with $\lambda_t = 0$. Considering the compatibility condition of Eq.(9), we have

$$[U_t, \phi] + [U, [V, \phi]] - [V_x, \phi] - [V, [U, \phi]] = 0. \tag{10}$$

The above equation, by virtue of Jacobi identity, can be transformed into

$$[U_t, \phi] - [V_x, \phi] + [[U, V], \phi] = 0, \tag{11}$$

from which, in view of the arbitrariness of the function $\phi$, we can obtain the zero-curvature equation

$$U_t - V_x + [U, V] = 0. \tag{12}$$

Given a new isospectral problem

$$\phi_x = U\phi, \quad \phi_t = V\phi, \quad \lambda_t = 0,$$

where

$$\begin{aligned} U =& U(u, \lambda) \\ =& e_1(1) + re_2(0) + \alpha s e_3(1) \\ &+ \beta s e_3(0) + u e_4(0) + v e_5(0), \\ V =& \sum_{n \ge 0} (a_n e_1(-n) + b_n e_2(-n) \\ &+ c_n(\alpha e_3(-n+1) + \beta e_3(-n)) \\ &+ d_n e_4(-n) + f_n e_5(-n)), \end{aligned}$$

the direct computation for the stationary zero-curvature equation

$$V_x = [U, V]$$

yields the recursion relations as follows:

$$\begin{cases} a_{nx} = \alpha(rc_{n+1} - sb_{n+1}) + \beta(rc_n - sb_n), \\ b_{nx} = 2b_{n+1} - 2ra_n, \\ c_{nx} = 2sa_n - 2c_{n+1}, \\ d_{nx} = d_{n+1} - ua_n - vb_n + rf_n, \\ f_{nx} = \alpha(sd_{n+1} - uc_{n+1}) + \beta(sd_n - uc_n) - f_{n+1} - va_n. \end{cases} \tag{13}$$

Taking $a_0 = 1$, $b_0 = c_0 = d_0 = f_0 = 0$ as the initial values and the additional initial conditions $a_j|_{[u]=0} = 0$, $(j \ge 1)$, along with $b_j|_{[u]=0} = c_j|_{[u]=0} = d_j|_{[u]=0} = f_j|_{[u]=0} = 0$, $(j \ge 0)$, where $[u] = (u, \partial u, \partial^{-1}u, \ldots)$, $a_i$, $b_i$, $c_i$, $d_i$, and $f_i$, are uniquely determined by Eq.(13) as follows:

$$a_1 = -\frac{1}{2}\beta rs, \quad b_1 = r, \quad c_1 = s,$$
$$d_1 = u, \quad f_1 = -v, \ \ldots$$

A direct calculation shows that

$$\begin{aligned} (\lambda^n V)_{+x} - [U, (\lambda^n V)_+] =& \alpha(rc_{n+1} - sb_{n+1})e_1(0) + 2b_{n+1}e_2(0) - 2c_{n+1}(\alpha\lambda + \beta)e_3(0) \\ &+ d_{n+1}e_4(0) + (\alpha(sd_{n+1} - uc_{n+1}) - f_{n+1})e_5(0). \end{aligned}$$

Let

$$V^{(n)} = (\lambda^n V)_+ + [-a_n + \alpha\partial^{-1}(rc_n - sb_n)]e_1(0),$$

from which, considering the zero-curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0,$$

we have

$$u_t = \begin{pmatrix} r \\ s \\ u \\ v \end{pmatrix}_t = \begin{pmatrix} 2b_{n+1} + 2\beta r\partial^{-1}(sb_{n+1} - rc_{n+1}) \\ -2c_{n+1} + 2\beta s\partial^{-1}(sb_{n+1} - rc_{n+1}) \\ d_{n+1} \\ \alpha(sd_{n+1} - uc_{n+1}) - f_{n+1} \end{pmatrix} = \tilde{J} \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ d_{n+1} \\ f_{n+1} \end{pmatrix}, \tag{14}$$

where

$$\tilde{J}=\begin{pmatrix} -2\beta r\partial^{-1}r & 2+2\beta r\partial^{-1}s & 0 & 0\\ -2-2\beta s\partial^{-1}r & 2\beta s\partial^{-1}s & 0 & 0\\ 0 & 0 & 1 & 0\\ -\alpha u & 0 & \alpha s & -1 \end{pmatrix}.$$

Furthermore, from Eq.(14), we can obtain the following recursion equations:

$$\begin{pmatrix} c_{n+1}\\ b_{n+1}\\ d_{n+1}\\ f_{n+1} \end{pmatrix}=\tilde{L}\begin{pmatrix} c_n\\ b_n\\ d_n\\ f_n \end{pmatrix},\quad \tilde{L}=\begin{pmatrix} L_{11} & 0\\ L_{21} & L_{22} \end{pmatrix},$$

with

$$L_{11}=\begin{pmatrix} -\frac{1}{2}\alpha\partial+\alpha s\partial^{-1}r-\frac{1}{2}\beta s\partial^{-1}r\partial & -\alpha s\partial^{-1}s-\frac{1}{2}\beta s\partial^{-1}r\partial\\ \alpha r\partial^{-1}r-\frac{1}{2}\beta r\partial^{-1}r\partial & -\frac{1}{2}\alpha\partial-\alpha r\partial^{-1}s-\frac{1}{2}\beta r\partial^{-1}s\partial \end{pmatrix},$$

$$L_{21}=\begin{pmatrix} \frac{1}{2}\beta u\partial^{-1}r\partial-\alpha u\partial^{-1}r & \alpha u\partial^{-1}s+\frac{1}{2}\beta u\partial^{-1}s\partial+v\\ \frac{1}{2}\alpha u\partial-\beta u-\alpha v\partial^{-1}r+\frac{1}{2}\beta v\partial^{-1}r\partial & \alpha sv+\alpha v\partial^{-1}s+\frac{1}{2}\beta v\partial^{-1}s\partial \end{pmatrix},$$

$$L_{22}=\begin{pmatrix} \partial & -r\\ \alpha s\partial+\beta s & -\alpha rs-\partial \end{pmatrix}.$$

Therefore, Eq.(14) can be rewritten as

$$u_t=\begin{pmatrix} r\\ s\\ u\\ v \end{pmatrix}_t=\tilde{J}\tilde{L}\begin{pmatrix} c_n\\ b_n\\ d_n\\ f_n \end{pmatrix}=\tilde{J}\tilde{L}^n\begin{pmatrix} r\\ s\\ u\\ -v \end{pmatrix}. \tag{15}$$

It is easy to verify that Eq.(15) is integrable because it is derived from the zero-curvature equations (12). Comparing $J$, $L$ and $\tilde{J}$, $\tilde{L}$ in Eqs.(7) and (15), and considering the definition of the integrable coupling,[3] we obtain that Eq.(15) is integrable coupling of Eqs.(7).

## 4. Concluding remarks

An integrable coupling of the coupled AKNS–Kaup–Newell hierarchy has been presented by enlarging the associated loop Lie algebra $\tilde{A}_1$ through spectral matrixes into $\tilde{A}_2$. Certainly, we can enlarge the loop Lie algebra $\tilde{A}_1$ into $\tilde{A}_3$ or $\tilde{A}_4$, also, can enlarge the loop Lie algebra $\tilde{A}_2$ into $\tilde{A}_3$ or $\tilde{A}_4$, and so on, which only make the spectral matrix $U$and $V$ turn into other forms, which, actually, enlarge the spectral matrix $U$ and $V$ i.e. enlarge the spectral problems in Ref.[14], which can be illustrated as follows:

$$\begin{pmatrix} U & U_1\\ 0 & 0 \end{pmatrix}\leftarrow(\text{or})\,U(\text{or})\rightarrow\begin{pmatrix} U & U_1\\ 0 & U_2 \end{pmatrix},$$

where $U$ is a square spectral matrix, $U_i$, $1\le i\le 2$, are suitably chosen order matrix. Especially, the technique adopted in Ref.[14] to construct the integrable couplings of various soliton equations seems much simpler, but it can present some larger integrable systems with quite shapely structures. Further, there arise a considerable number of problems concerning whether the resulting systems possess other excellent properties,[15,16] such as Hamiltonian pairs and Liouville integrability, etc,[17–22] because, to a certain ex-

tent, the subsistent coupling hierarchies usually have anomalous features. In addition, can the method be used for discussing integrable couplings for discrete systems? These are matters awaiting investigation stage by stage.

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