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Chaotic coupling synchronization of hyperchaotic oscillators*

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In this paper, two kinds of chaotic coupling synchronization schemes are presented. The synchronizability of the coupled hyperchaotic oscillators is proved mathematically and the numerical simulation is also carried out. The numerical calculation of the largest conditional Lyapunov exponent shows that in a given range of coupling strengths, chaotic-coupling synchronization is quicker than the typical continuous-coupling synchronization.

Keywords: chaos synchronization, hyperchaotic oscillator, chaotic coupling
PACC: 0545

1. Introduction

Since the first presentation of the now-well-known chaos control method[1] and drive-response chaos synchronization scheme,[2,3] the chaos control and synchronization have received increasing attention due to their predictable potentials in technological applications.[4−8] A number of chaos synchronization schemes have been put forward. Among them the most popular one is continuous-coupling synchronization scheme,[2−4,9,10] where chaos systems are coupled to each other permanently so that the synchronization errors can be controlled within a given tolerance or converge to zero asymptotically. Other well-known chaos synchronization schemes include, for instance, some impulse synchronization algorithms, in which the drive signals are intermittently transmitted to the response system.[11−13]

Many hyperchaotic oscillators have been proposed and designed in the past; their examples include the one reported in Ref.[14], based on which a unidirectional continuous-coupling synchronization scheme and a periodic impulse synchronization algorithm with only one single variable connection have been designed lately.[15] In this paper, synchronization of two hyperchaotic oscillators via unidirectional chaotic-coupling is studied. We give a mathematical proof of the synchronizability of unidirectional chaotic-coupling with a single-variable connection and then the numerical simulation results. Finally, we also make a comparison of convergence speeds between a chaotic-coupling synchronization scheme and a continuous-coupling synchronization scheme in terms of the numerical calculation of the largest conditional Lyapunov exponent.

2. Hyperchaotic oscillators

A simple hyperchaotic oscillator constructed in Ref.[14] is shown in Fig.1. This circuit includes a combined parallel–series LC circuit, $L_1C_1 - L_2C_2$, and an OpAmp OA with resistors R_1, R_3 and R_4, which plays the roles of both the negative impedance converter (NIC) and the output amplifier. The oscillator is described by the following equations:

$$\begin{aligned} C_1\frac{\mathrm{d}V_{C_1}}{\mathrm{d}t} &= \frac{V_{C_1}}{R_1} - I_{L_1} - I_{L_2}, \\ L_1\frac{\mathrm{d}I_{L_1}}{\mathrm{d}t} &= V_{C_1}, \\ L_2\frac{\mathrm{d}I_{L_2}}{\mathrm{d}t} &= V_{C_1} - V_{C_2}, \\ C_2\frac{\mathrm{d}V_{C_2}}{\mathrm{d}t} &= I_{L_2} - \frac{V_{C_2} - V_0}{R_2}H(V_{C_2} - V_0), \end{aligned} \tag{1}$$

where V_{C_1} and V_{C_2} denote the voltages across the capacitances C_1 and C_2, respectively, I_{L_1} and I_{L_2} represent the currents flowing through the inductances L_1

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and L_2, respectively, V_0 is the forward voltage drop across the diode, and $H(u)$ is the Heaviside function

$$H(u) = \begin{cases} 0 & u < 0, \\ 1 & u \geq 0. \end{cases} \tag{2}$$

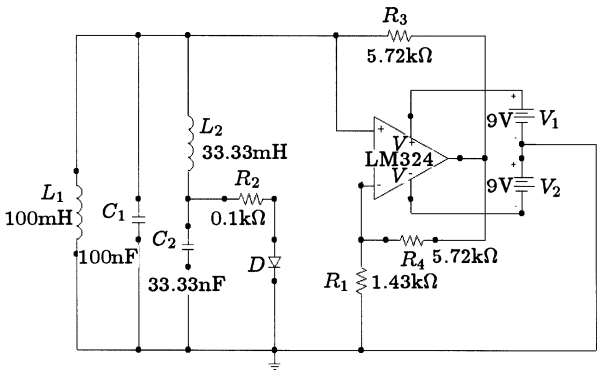


Fig.1. The hyperchaotic oscillator.

When the circuit components take the values shown in Fig.1, the circuit displays hyperchaos,[14] as shown in Fig.2.

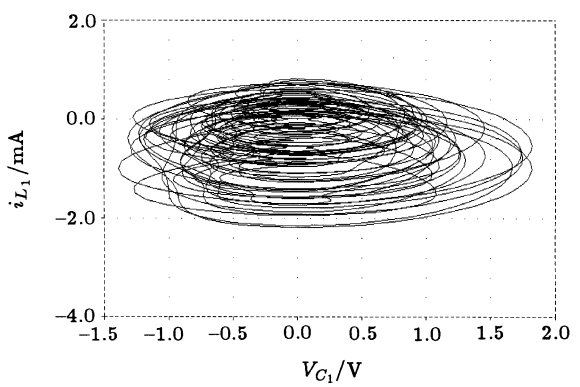


Fig.2. The simulation result of the circuit hyperchaos.

To simplify the analysis, we introduce the following notation:

$$\begin{aligned} &V_{C_1}/V_0 = x, \quad \rho I_{L_1}/V_0 = y, \quad \rho I_{L_2}/V_0 = z, \\ &V_{C_2}/V_0 = \nu, \quad \rho/\tau = \theta, \quad \rho = \sqrt{L_1/C_1}, \quad \tau = \sqrt{L_1 C_1}, \\ &a = \rho/R_1, \quad b = \rho/R_2, \quad c = L_1/L_2, \quad d = C_1/C_2, \end{aligned}$$

and let $\dot{u} = \mathrm{d}u/\mathrm{d}\theta$. Then, the circuit equation becomes

$$\begin{aligned} \dot{x} &= ax - y - z, \\ \dot{y} &= x, \\ \dot{z} &= c(x - v), \\ \dot{v} &= d\left[z - b(v-1)H(v-1)\right] . \end{aligned} \tag{3}$$

When a= 0.7, b= 10, $c = d$= 3 (the same as the values shown in Fig.(1)), the calculated Lyapunov exponent is (0.123, 0.067, 0.00, −9.75), the same as that reported in Ref.[14]. Since the system has two positive Lyapunov exponents, it is hyperchaotic.

3. Chaotic-coupling synchronization schemes

In this section, two kinds of chaotic-coupling synchronization schemes A and B are presented. Then, the synchronizability of the chaotic-coupling synchronization scheme A is proved mathematically.

3.1. Chaotic-coupling synchronization schemes A and B

When Scheme A is applied, the response system of the system (3) is described as

$$\begin{aligned} \dot{x}' &= ax' - y' - z' + H(v-1)\cdot k(x - x'), \\ \dot{y}' &= x', \\ \dot{z}' &= c(x' - v'), \\ \dot{v}' &= d\left[z' - b(v'-1)H(v'-1)\right], \end{aligned} \tag{4}$$

where k is the coupling coefficient, $H(x)$ is the Heaviside function defined by Eq.(2).

It follows from Eq.(4) that when $v \geq 1$, namely, $V_{C_2} \geq V_0$, the diode in drive circuit is on , and$H(v-1) = 1$, so the driving signal x is transmitted ; while $v < 1$, namely, the diode in drive circuit is off, and $H(v-1) = 0$, thus the response system (4) evolves freely.

When Scheme B is applied, whether the drive signal being transmitted is just opposite to that of scheme A, the response system of the system (3) is given by

$$\begin{aligned} \dot{x}' &= ax' - y' - z' + H(1-v)\cdot k(x - x'), \\ \dot{y}' &= x', \\ \dot{z}' &= c(x' - v'), \\ \dot{v}' &= d\left[z' - b(v'-1)H(v'-1)\right] . \end{aligned} \tag{5}$$

According to Eqs.(4) and (5), whether the drive signal being transmitted depends on the state of diode in the drive circuit working in a hyperchaotic state, both synchronization scheme A and scheme B are chaotically coupled.

3.2. Mathematical proof of synchronization scheme A

When the synchronization scheme A is applied, the synchronization error system is described as follows:

$$\begin{aligned}\dot{e}_1 &= ae_1 - e_2 - e_3 - k \cdot H(v-1) \cdot e_1,\\ \dot{e}_2 &= e_1,\\ \dot{e}_3 &= ce_1 - ce_4,\\ \dot{e}_4 &= de_3 - db(v-1)H(v-1)\\ &\quad +db(v'-1)H(v'-1),\end{aligned} \tag{6}$$

where $e_1 = x - x'$, $e_2 = y - y'$, $e_3 = z - z'$, and $e_4 = v - v'$.

When the synchronization scheme B is applied, the synchronization error system is given by

$$\begin{aligned}\dot{e}_1 &= ae_1 - e_2 - e_3 - k \cdot H(1-v) \cdot e_1,\\ \dot{e}_2 &= e_1,\\ \dot{e}_3 &= ce_1 - ce_4,\\ \dot{e}_4 &= de_3 - db(v-1)H(v-1)\\ &\quad +db(v'-1)H(v'-1),\end{aligned} \tag{7}$$

where $e_1 = x - x'$, $e_2 = y - y'$, $e_3 = z - z'$, and $e_4 = v - v'$.

In the following, two theorems are given and the theorem 1 is proved based on the Lyapunov stabilization theory.

Theorem 1 If the coupling coefficient k satisfies $k > \dfrac{a}{p}$, then the error dynamical system (6) is globally asymptotically stable about the origin, implying that the two chaotic systems (3) and (4) are asymptotically synchronized with each other, where a is a system parameter of the system (3) and p denotes the probability for the occurrence of the event $v \geq 1$.

Theorem 2 If the coupling coefficient k satisfies $k > \dfrac{a}{1-p}$, then the error dynamical system (7) is globally asymptotically stable about the origin, implying that the two chaotic systems (3) and (5) are asymptotically synchronized with each other, where a and p have the same meanings as those in Theorem 1, respectively.

For the sake of concision, we only present the proof of Theorem 1. The proof of Theorem 2 is given in the Appendix.

Proof of Theorem 1:

Construct a Lyapunov function for the error system (6) as follows

$$\Delta = \frac{1}{2}(ce_1^2 + ce_2^2 + e_3^2 + e_4^2) \geq 0, \tag{8}$$

which is zero if and only if $e_1 = e_2 = e_3 = e_4 = 0$.

The derivative of Δ with respect to time is given by

$$\dot{\Delta} = ce_1\dot{e}_1 + ce_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4. \tag{9}$$

Substituting Eq.(6) into Eq.(9) yields

$$\begin{aligned}\dot{\Delta} =& c(a - k \cdot H(v-1))e_1^2 - e_4\\ &\cdot d[b(v-1)H(v-1) - b(v'-1)H(v'-1)],\\ \dot{\Delta} =& c(a - k \cdot H(v-1))e_1^2 - e_4 \cdot d \cdot M,\end{aligned} \tag{10}$$

where

$$M = b(v-1)H(v-1) - b(v'-1)H(v'-1), \tag{11}$$

namely,

$$M = \begin{cases} be_4 & v \geq 1, v' \geq 1,\\ b(v-1) & v \geq 1, v' < 1,\\ -b(v'-1) & v < 1, v' \geq 1,\\ 0 & v < 1, v' < 1.\end{cases} \tag{12}$$

Since $e_4 = v - v'$, one obtains

$$-e_4 \cdot d \cdot M = \begin{cases} -bde_4^2 \leq 0 & v \geq 1, v' \geq 1,\\ -(v-v') \cdot d \cdot b(v-1) \leq 0 & v \geq 1, v' < 1,\\ (v-v') \cdot d \cdot b(v'-1) \leq 0 & v < 1, v' \geq 1,\\ 0 & v < 1, v' < 1.\end{cases} \tag{13}$$

Based on the above analysis, one has $-e_4 \cdot d \cdot M \leq 0$ in all cases.

When $v \geq 1$, then $H(v-1) = 1$, and Eq.(10) becomes

$$\dot{\Delta} = c(a-k)e_1^2 - e_4 \cdot d \cdot M. \tag{14}$$

When $v < 1$, then $H(v-1) = 0$, and Eq.(10) becomes

$$\dot{\Delta} = c \cdot a \cdot e_1^2 - e_4 \cdot d \cdot M. \tag{15}$$

Because the drive circuit works in the hyperchaotic state, the diode in the drive circuit switches on and off randomly, that is, the state variable v takes values randomly in an attracting field. Given that the probability for the occurrence of the event $v \geq 1$ is p, then the probability for the occurrence of the event $v < 1$ is $1 - p$, so one has

$$E\dot{\Delta} = \dot{\Delta}\Big|_{v \geq 1} \cdot p + \dot{\Delta}\Big|_{v<1} \cdot (1-p), \tag{16}$$

where $E\dot{\Delta}$ denotes the average value of $\dot{\Delta}$. Substituting Eq.(14) and Eq.(15)into Eq.(16), one obtains

$$E\dot{\Delta} = (c(a-k)e_1^2 - e_4 \cdot d \cdot M) \cdot p + (c \cdot a \cdot e_1^2 - e_4 \cdot d \cdot M) \cdot (1-p). \quad (17)$$

Simplifying (17), one has

$$E\dot{\Delta} = c(a-k)e_1^2 \cdot p + c \cdot a \cdot e_1^2(1-p) - e_4 \cdot d \cdot M. \quad (18)$$

Since $-e_4 \cdot d \cdot M \leq 0$, we have

$$E\dot{\Delta} \leq c(a-k)e_1^2 \cdot p + c \cdot a \cdot e_1^2 \cdot (1-p) = c(a - k \cdot p)e_1^2.$$

When $k > \dfrac{a}{p}$, $E\dot{\Delta} < 0$. From Eq.(10), we know $\dot{\Delta} = 0$ when $\Delta = 0$.

According to the Lyapunov theory, the error system (6) is globally asymptotically stable about the origin $e_1 = e_2 = e_3 = e_4 = 0$, namely, when $k > \dfrac{a}{p}$, $t \to \infty$, the systems (3) and (4) will achieve completely synchronization with each other. Note that $k > \dfrac{a}{p}$is only a sufficient but not necessary condition for the synchronization of the systems (3) and (4) with each other.

Theorem 2 is proved by analogy. It can be seen in the Appendix. With numerical study, one obtains the probability for the occurrence of the event "the diode in drive circuit is on" being 0.3278, namely, $P(v \geq 1) = 0.3278$, sequentially, the synchronization thresholds of the scheme A and the scheme B are $k > 2.1354$ and $k > 1.0414$, respectively.

4. Numeric simulation

For verifying the correctness of the chaotic-coupling synchronization schemes, we set up a simulation model of chaotic-coupling hyperchaotic oscillators using Simulink in Matlab software. Figures (3) and (4) show the graphs of the mean square error versus time by applying the schemes A and B, respectively. The vertical coordinates in Fig.3 and Fig.4 denote e_m, where $e_m = \sqrt{e_1^2 + e_2^2 + e_3^2 + e_4^2/4}$ is the mean square error. Figure (5) shows the diagram of state variable x in Eq.(3) versus x' in Eq.(4) with the synchronization A applied and Fig.(6) shows the diagram of state variable y in Eq.(3) versus y' in Eq.(5) with the scheme B applied.

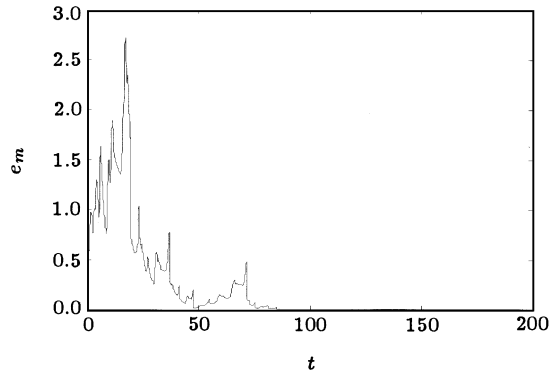


Fig.3. Synchronization error curve with the scheme A and $k = 35$ applied.

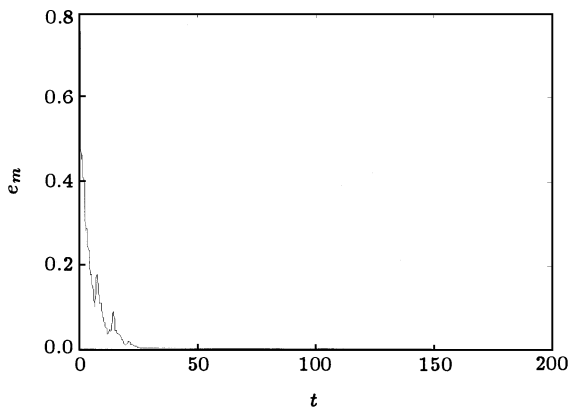


Fig.4. Synchronization error curve with the scheme B and $k = 3.2$ applied.

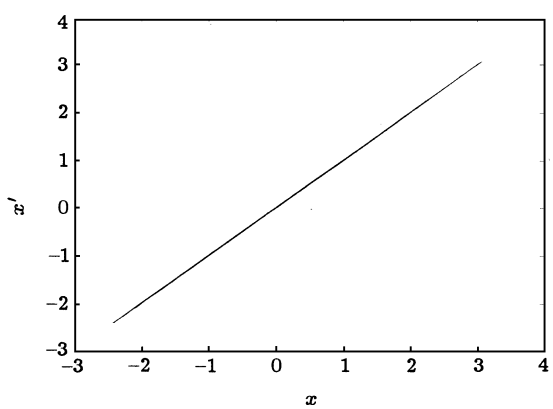


Fig.5. Relationship between x and x', where x and x' are state variables in Eq.(3) and Eq.(4), respectively.

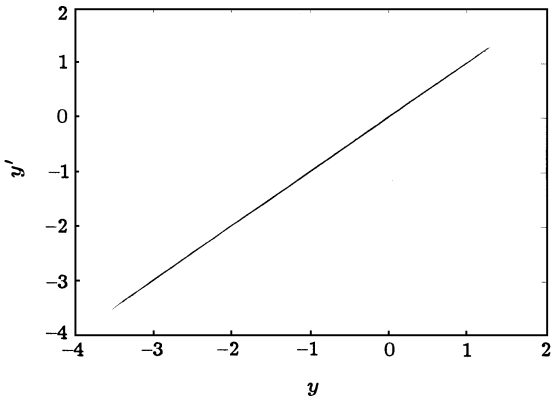


Fig.6. Relationship between y and y', where y and $y'$are state variables in Eq.(3) and Eq.(5), respectively.

It can be seen from the above simulation results that the proposed chaotic coupling synchronization schemes are indeed effective and complete synchronization between two coupled hyperchaotic oscillators can be achieved.

5. Comparison of three synchronization schemes

If the Heaviside function $H(v\text{-}1)$ is ignored in the first subsystem of the response system (4), then the drive signal is transmitted permanently, which is a continuous-coupled synchronization scheme proposed in Ref.[15] (referred to as the scheme C below). Since the largest conditional Lyapunov exponent denotes the average convergence or divergence speed of synchronization error, the smaller the negative largest conditional Lyapunov exponent, the shorter the response time. To compare the convergence speeds of the three synchronization schemes A, B and C, which are applied to two coupled hyperchaotic oscillators, the largest conditional Lyapunov exponents of these three synchronization schemes under different values of coupling strength k in a range of [1,1000], are calculated. The calculations are shown in Fig.7, where the symbols ◯, □ and ▽ denote the results with the synchronization schemes C, B and A applied, respectively. Numerical results show when $1 < k \leq 5$ Scheme C is the fastest one among the three in achieving the synchronization; when $5 < k \leq 20$, Scheme B is the fastest; when $20 < k \leq 1000$, Scheme A is the fastest.

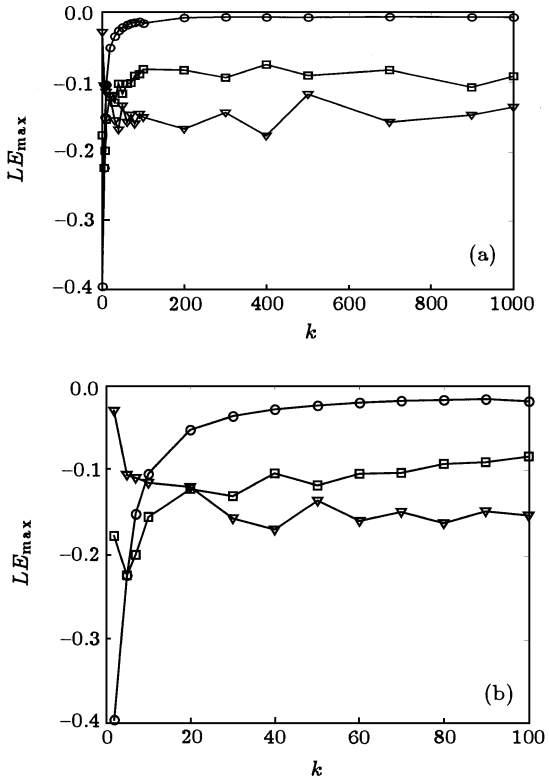


Fig.7. The largest conditional Lyapunov exponents of the three synchronization schemes A, B and C under different values of coupling strength k in the range [1, 1000] (◯ for the result with Scheme C applied; □ for the result with Scheme B applied; ▽ for the result with Scheme A applied).

6. Conclusion

This paper has investigated two unidirectional chaotic coupling synchronization schemes with single-variable connection for hyperchaotic oscillators. At first, the synchronizability of chaotic-coupling synchronization schemes is proved mathematically. Then the numerical simulation is carried out to verify the correctness of proposed synchronization schemes. Finally, the convergence speeds of the two chaotic-coupling synchronization schemes and a continuous-coupling synchronization scheme are compared in terms of the largest conditional Lyapunov exponent under different values of coupling strength k. It is found that the chaotic-coupling synchronization scheme is quicker than the continuous-coupling one to achieve the synchronization in a given range of k. The studies presented here can provide some insights and guidelines for choosing the fast convergent schemes for synchronizing the hyperchaotic oscillators with each other.

Appendix

Proof of Theorem 2:

Construct a Lyapunov function for the error system (7) as follows:

$$\Delta = \frac{1}{2}(ce_1^2 + ce_2^2 + e_3^2 + e_4^2) \geq 0, \tag{A1}$$

which is zero if and only if $e_1 = e_2 = e_3 = e_4 = 0$.

The derivative of Δ with respect to time is given by

$$\dot{\Delta} = ce_1\dot{e}_1 + ce_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4. \tag{A2}$$

Substituting Eq.(7) into Eq.(A2) yields

$$\begin{aligned}\dot{\Delta} =& c(a - k \cdot H(1-v))e_1^2 - e_4 \\ & \cdot d[b(v-1)H(v-1) - b(v'-1)H(v'-1)], \\ \dot{\Delta} =& c(a - k \cdot H(1-v))e_1^2 - e_4 \cdot d \cdot M. \end{aligned} \tag{A3}$$

According to Eqs.(12) and (13) in this paper, we have $-e_4 \cdot d \cdot M \leq 0$ in all cases.

When $v > 1$, $H(1-v) = 0$, and

$$\dot{\Delta} = c \cdot a \cdot e_1^2 - e_4 \cdot d \cdot M. \tag{A4}$$

When $v \leq 1$, $H(1-v) = 1$, and

$$\dot{\Delta} = c(a-k)e_1^2 - e_4 \cdot d \cdot M. \tag{A5}$$

Because the drive circuit works in the hyperchaotic state, then the measure of $v = 1$ is zero in an attracting field, the probability for the occurrence of the event $v \geq 1$ is equal to that of the event $v > 1$. Given that the probability for the occurrence of the event $v \geq 1$ is p, then the probability for the occurrence of the event $v < 1$ is $1 - p$, and one has

$$E\dot{\Delta} = \dot{\Delta}\Big|_{v>1} \cdot p + \dot{\Delta}\Big|_{v\leq 1} \cdot (1-p), \tag{A6}$$

where $E\dot{\Delta}$ is the average value of $\dot{\Delta}$.

Substituting Eqs.(A4) and (A5) into Eq.(A6) yields

$$\begin{aligned} E\dot{\Delta} =& (c \cdot a \cdot e_1^2 - e_4 \cdot d \cdot M) \cdot p + (c(a-k)e_1^2 \\ & - e_4 \cdot d \cdot M) \cdot (1-p). \end{aligned} \tag{A7}$$

Simplifying (A7), one obtains

$$E\dot{\Delta} = c \cdot a \cdot e_1^2 \cdot p + c(a-k)e_1^2 \cdot (1-p) - e_4 \cdot d \cdot M. \tag{A8}$$

Since $-e_4 \cdot d \cdot M \leq 0$, we have

$$\begin{aligned} E\dot{\Delta} \leq& c \cdot a \cdot e_1^2 \cdot p + c(a-k)e_1^2 \cdot (1-p) \\ =& c \cdot (a - (1-p) \cdot k) \cdot e_1^2. \end{aligned} \tag{A9}$$

When $k > \dfrac{a}{1-p}$, we have $E\dot{\Delta} < 0$. From Eq.(A2), we know $\dot{\Delta} = 0$ when $\Delta = 0$. According to the Lyapunov stabilization theory, the error system (7) is globally asymptotically stable about the origin $e_1 = e_2 = e_3 = e_4 = 0$, namely, when $k > \dfrac{a}{1-p}$, $t \to \infty$, the systems (3) and (5) will achieve completely synchronization with each other. Note that $k > \dfrac{a}{1-p}$ is only a sufficient but not necessary condition for the synchronization of systems (3) and (5) with each other.

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