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# New exact excitations and soliton fission and fusion for the (2+1)-dimensional Broer–Kaup–Kupershmidt system*

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With the help of an extended mapping approach, a series of new types of exact excitations with two arbitrary functions of the (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) system is derived. Based on the derived solitary wave excitation, some specific soliton fission and fusion solutions of the higher-dimensional BKK system are also obtained.

**Keywords:** extended mapping approach, BKK system, exact solution, soliton fission, soliton fusion
**PACC:** 0340, 0200, 0365

## 1. Introduction

It is well known that the interaction between soliton solutions of integrable models is considered to be completely elastic. That is to say, the amplitude, velocity and wave shape of a soliton are not changed after nonlinear interaction.[1] However, for some specific solutions of some (2+1)-dimensional models, the interactions among solitonic excitations like peakons and compactons are not completely elastic since their shapes are changed after their collisions.[2] Furthermore, for some (1+1)-dimensional models, two or more solitons may fuse into one soliton at a specific time while sometimes one soliton may fission into two or more solitons at another specific time.[3] These phenomena are often called soliton fusion and soliton fission respectively. These soliton fusion and fission phenomena have been observed in many physical systems such as organic membrane and macromolecular material,[4] and many physical fields like plasma physics, nuclear physics and hydrodynamics.[5] Recently, Zhang *et al*[6] and Lin *et al*[7] have studied the evolutions of soliton solutions for two (1+1)-dimensional PDEs with time and revealed the solitons fission and soliton fusion phenomena. Wang *et al*[8] further discussed two (1+1)-dimensional models, the Burgers equation and the Sharma–Tasso–Olver (STO) equation, via Hirota's direct method and the Bäcklund transformation, and also found the soliton fission and soliton fusion phenomena. Now an interesting and important problem is whether we can obtain the soliton fission and fusion solutions in higher-dimensional systems. The main purpose of this paper is to search for these significant soliton fission and soliton fusion phenomena in the following celebrated (2+1)-dimensional Broer–Kaup–Kupershmidt (BKK) system:[9]

$$H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0,$$
$$G_t + 2(GH)_x + G_{xx} = 0. \tag{1}$$

The BKK system (1) is used to model the nonlinear and dispersive long gravity waves travelling in two horizontal directions in shallow water with uniform depth, and it can be also derived from the celebrated Kadomtsev–Petviashvili (KP) equation by the symmetry constraint.[10] When $y = x$, the (2+1)-dimensional BKK system is reduced further to a usual (1+1)-dimensional BKK system, which can be used to describe the propagation of long waves in shallow water.[11] Using some suitable dependent and independent variable transformations, Chen and Li[12] have

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proved that the (2+1)-dimensional BKK system can be transformed to the (2+1)-dimensional dispersive long wave equation (DLWE) and (2+1)-dimensional Ablowitz–Kaup–Newell–Segur (AKNS) system. The (2+1)-dimensional BKK system has been widely investigated in detail by many researchers.[13] However, to the best of our knowledge, its soliton fission and soliton fusion phenomena have not been reported in the preceding literature.

As is well known, in order to search for the solitary wave solutions to a nonlinear physical model, we can apply different approaches. One of the most efficient methods of finding soliton excitations of a physical model is the so-called mapping transformation method.[14,15] With the mapping transformation idea and based on the general reduction theory,[16] we extend the mapping approach. The basic idea of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with independent variables $x = (x_0 = t, x_1, x_2, \cdots, x_m)$ and dependent variable $u$,it may be in the form

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \cdots) = 0, \tag{2}$$

where $P$ is in general a polynomial function of some arguments, and the subscripts denote the partial derivatives. We assume that its solution is in an extended symmetry form, namely

$$u = \sum_{i=-n}^{n} \alpha_i(x)\phi^i(\omega(x)), \tag{3}$$

with

$$\phi' = \sigma + \phi^2, \tag{4}$$

where $\alpha_i(x)$, $\omega(x)$ are arbitrary functions to be determined, $x = (x_0 = t, x_1, x_2, \cdots, x_m)$, $\sigma$ is a constant and the $\phi'$ denotes the first derivative of function $\phi$ with respect to $\omega$. To determine $u$ explicitly, one can use the following procedure: first, determining $n$ in an approach similar to the usual mapping approach by balancing the highest nonlinear term and the highest-order partial term in the given NPDE; second, substituting Eqs.(3) and (4) into the given NPDE and collecting coefficients of polynomials of $\phi$, then eliminating each coefficient to construct a set of partial differential equations of $\alpha_i(x)(i = -n, ..., -1, 0, 1, ..., n)$ and $\omega(x)$; third, solving the system of partial differential equations to obtain $\alpha_i(x)$ and $\omega(x)$. Finally, as Eq.(4) possesses the general solution

$$\phi = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\omega), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\omega), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}\omega), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\omega), & \sigma > 0, \\ \dfrac{-1}{\omega}, & \sigma = 0, \end{cases} \tag{5}$$

substituting $\alpha_i(x)$, $\omega(x)$ and Eq.(5) into Eq.(3), one can derive the exact solutions of the given NPDE.

## 2. New exact solutions of the (2+1)-dimensional BKK system

First, let us make a transformation for Eq.(1): $G = H_y$. Substituting the transformation into Eq.(1) yields

$$H_{yt} + 2(H_x H)_y + H_{xxy} = 0. \tag{6}$$

Now we apply the extended mapping approach to Eq.(6). By using the balancing procedure, the ansatz (3) becomes

$$H = f + g\phi(q) + h\phi^{-1}, \tag{7}$$

where $f, g, h$ and $q$ are arbitrary functions of $\{x, y, t\}$ to be determined. Substituting Eqs.(7) and (4) into Eq.(6) and collecting coefficients of polynomials of $\phi$, then setting each coefficient to zero, we have

$$6gq_yq_x(q_x + g) = 0, \tag{8}$$

$$\begin{aligned} &4q_yq_x(g + fg + g_x) + 2q_{xy}(2q_x + g^2) \\ &+ q_y(4g_xg + 2gq_{xx} + 2gq_t) + 2g_yq_x^2 = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} &8q_yq_xg\sigma(g + q_x) + 2q_{xy}(fg + g + g_x) \\ &+ q_y(2fg_x + g_t + g_{xx} + 2f_xg) \\ &+ q_x(2g_{xy} + 2gf_y + 2fg_y) + gq_{yt} \\ &+ 2g_xg_y + gq_{xxy} + g_y(q_t + q_{xx}) = 0, \end{aligned} \tag{10}$$

$$\begin{aligned} &2g\sigma q_{xy}(2q + g) + 4\sigma q_xq_y(fg + g + g_x) \\ &+ 2gq_y\sigma(q_t + 2q_x + q_{xx}) + g_{yt} + g_{xxy} + 2g_yq_x^2\sigma \\ &+ 2f_xg_y + 2gf_{xy} + 2g_xf_y + 2fg_{xy} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &4hq_yq_x\sigma(f + 1) + 2hq_{xy}(2q_x\sigma - h) \\ &+ 2hq_y\sigma(q_t + q_{xx}) + 2q_xh_y(q_x\sigma - 2h) + 2f_xh_y \\ &+ h_{2xy} - 4hh_xq_y + h_{yt} + 2hf_{xy} \\ &+ 2h_xf_y + 2fh_{xy} = 0, \end{aligned} \tag{12}$$

$$q_{xy}\sigma(-h-2fh-2h_x)+8q_xq_y\sigma(h^2-q_x\sigma)$$
$$-q_y\sigma(2f_xh+2fh_x+h_t+h_{xx})$$
$$-q_x\sigma(2hf_y+2h_{xy}+2fh_y)-h\sigma q_{yt}-h_yq_{2x}\sigma$$
$$+2hh_{xy}+2h_xh_y-h_yq_t\sigma=0, \tag{13}$$

$$2hq_y\sigma^2(2fq_x+q_{2x})+4h_xq_y\sigma(q_x\sigma-h)$$
$$+2q_yhq_t\sigma^2+2hq_x^2\sigma^2+q_{xy}(4hq_x\sigma^2-2h^2\sigma)$$
$$+2h_yq_x^2\sigma^2-4hq_xh_y\sigma=0, \tag{14}$$

$$6hq_xq_y\sigma^2(h-q_x\sigma)=0, \tag{15}$$

$$2h_{xy}(g-2g_x)+2g_{xy}(h-2g_x\sigma)+q_{xxy}(g\sigma-h)$$
$$+q_{xx}(g_y\sigma-h_y)+q_{yt}(g\sigma-h)$$
$$+q_x(2fg_y\sigma+2gf_y\sigma-2hf_y-2fh_y)$$
$$+q_{xy}(2fg\sigma-2h_x-2fh+2g_x\sigma)$$
$$+g_y(q_t\sigma+2h_x)+h_y(2g_x-q_t)$$
$$+f_{yt}+f_{xxy}+2ff_{xy}+2f_xf_y$$
$$+q_y(2gq_x^2\sigma^2-2hq_x^2\sigma-2fh_x$$
$$+2g^2q_x\sigma^2+g_{xx}\sigma+g_t\sigma+2fg_x\sigma)$$
$$+2h^2q_x+2f_xg\sigma-h_t-2f_xh-h_{xx})=0. \tag{16}$$

Based on Eqs.(8) and (15), we have

$$g=-q_x,\quad h=q_x\sigma. \tag{17}$$

Reducing Eqs.(9)–(14), (16) by using Eq.(17), we obtain

$$f=-\frac{q_{xx}+q_t}{2q_x}, \tag{18}$$

with

$$(12q_{xx}^2q_t-4q_xq_{xxx}q_t-8q_xq_{xx}q_{xxx}$$
$$-4q_tq_xq_{xt}+q_x^2q_{tt}-8q_{xx}q_xq_{xt}+2q_x^2q_{xxt}+3q_{xx}q_t^2$$
$$+9q_{xx}^3-16\sigma q_{xx}q_x^4+q_x^2q_{xxxx})q_{xy}$$
$$+(2q_x^2q_{xxx}-4q_xq_{xx}^2-2q_tq_xq_{xx}+4q_x^2q_{xxy}+2q_x^2q_{xxx}$$
$$-2q_tq_xq_{xx}+2q_x^2q_{xt})q_{yt}+(2q_x^2q_t+4q_x^2q_{xx})q_{xyt}$$
$$-(9q_xq_{xx}^2+q_xq_t^2-4q_x^2q_{xxx}+4q_x^2q_{xt}$$
$$+16q_x^5\sigma+8q_xq_tq_{xx})q_{xxy}+(2q_tq_x^2+4q_{xx}q_x^2)q_{xxxy}$$
$$-q_x^3(q_{xxxxy}+2q_{xxyt}+q_{ytt})=0. \tag{19}$$

Substituting Eqs.(17), (18) and the solution of Eq.(19) into Eq.(7), we can obtain the solution of Eq.(6).

Obviously, it is very difficult to obtain the general solution of Eq.(19). Fortunately, in this special case, one of special solutions can be expressed as

$$q=\chi(x,t)+\varphi(y), \tag{20}$$

where $\chi\equiv\chi(x,t),\varphi\equiv\varphi(y)$ are two arbitrary variable-separable functions of $x,t$, and $y$ respectively. Based on the solutions of Eq.(4), one can obtain the exact solutions of Eq.(1).

**Case 1** For $\sigma<0$, we can derive the following solitary wave solutions of Eq.(1)

$$H_1=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{-\sigma}\chi_x\tanh(\sqrt{-\sigma}(\chi+\varphi))$$
$$-\frac{\sigma\chi_x}{\sqrt{-\sigma}\tanh(\sqrt{-\sigma}(\chi+\varphi))}, \tag{21}$$
$$G_1=\frac{[\tanh^2(\sqrt{-\sigma}(\chi+\varphi))-1]^2\sigma\chi_x\varphi_y}{\tanh^2(\sqrt{-\sigma}(\chi+\varphi))}, \tag{22}$$
$$H_2=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{-\sigma}\chi_x\coth(\sqrt{-\sigma}(\chi+\varphi))$$
$$-\frac{\sigma\chi_x}{\sqrt{-\sigma}\coth(\sqrt{-\sigma}(\chi+\varphi))}, \tag{23}$$
$$G_2=\frac{[\coth^2(\sqrt{-\sigma}(\chi+\varphi))-1]^2\sigma\chi_x\varphi_y}{\coth^2(\sqrt{-\sigma}(\chi+\varphi))}, \tag{24}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

**Case 2** For $\sigma>0$, we can obtain the following periodic wave solutions of Eq.(1):

$$H_3=-\frac{\chi_{xx}+\chi_t}{2\chi_x}-\sqrt{\sigma}\chi_x\tan(\sqrt{\sigma}(\chi+\varphi))$$
$$+\frac{\sigma\chi_x}{\sqrt{\sigma}\tan(\sqrt{\sigma}(\chi+\varphi))}, \tag{25}$$
$$G_3=-\frac{[\tan^2(\sqrt{\sigma}(\chi+\varphi))+1]^2\sigma\chi_x\varphi_y}{\tan^2(\sqrt{\sigma}(\chi+\varphi))}, \tag{26}$$
$$H_4=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\sqrt{\sigma}\chi_x\cot(\sqrt{\sigma}(\chi+\varphi))$$
$$-\frac{\sigma\chi_x}{\sqrt{\sigma}\cot(\sqrt{\sigma}(\chi+\varphi))}, \tag{27}$$
$$G_4=-\frac{[\cot^2(\sqrt{\sigma}(\chi+\varphi))+1]^2\sigma\chi_x\varphi_y}{\cot^2(\sqrt{\sigma}(\chi+\varphi))}, \tag{28}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

**Case 3** For $\sigma=0$, we can derive the following variable-separable solution of Eq.(1)

$$H_5=-\frac{\chi_{xx}+\chi_t}{2\chi_x}+\frac{\chi_x}{\chi+\varphi}, \tag{29}$$
$$G_5=-\frac{\chi_x\varphi_y}{(\chi+\varphi)^2}, \tag{30}$$

with two arbitrary functions being $\chi(x,t)$ and $\varphi(y)$.

## 3. Some specific examples: soliton fission and fusion solutions in the (2+1)-dimensional BKK system

In this section, we mainly discuss the solitary wave solutions, namely Case 1. Owing to the arbitrariness of the functions $\chi(x,t)$ and $\varphi(y)$ included in

this case, the physical quantities $H$ and $G$ may possess rich localized structures. For simplicity in the following discussion, we merely analyse the solitary wave excitation $G_1$ expressed by Eq.(22) and rewrite it in a simple form, namely

$$G \equiv -G_1 = -\sigma\chi_x\varphi_y \coth^2(\sqrt{-\sigma}(\chi+\varphi)) \times \operatorname{sech}^4(\sqrt{-\sigma}(\chi+\varphi)). \tag{31}$$

Generally, the interactions between solitons may be regarded as completely elastic. For instance, when $\chi(x,t)$ and $\varphi(y)$ are considered to be

$$\chi = 1 + 0.1\sec h(x+t) + 0.5\operatorname{sech}(x-t),$$
$$\varphi = 1 + 0.1\operatorname{sech}(y), \tag{32}$$

and $\sigma = -1$ in Eq.(31), we can obtain a solitary wave solution of Eq.(1) with elastic behaviour. Figure 1 shows an evolutional profile corresponding to the physical quantity $G$ expressed by Eq.(31). From Fig.1 and through detailed analysis, we find that the shapes, amplitudes and velocities of the two solitary waves are completely conserved after their interaction.

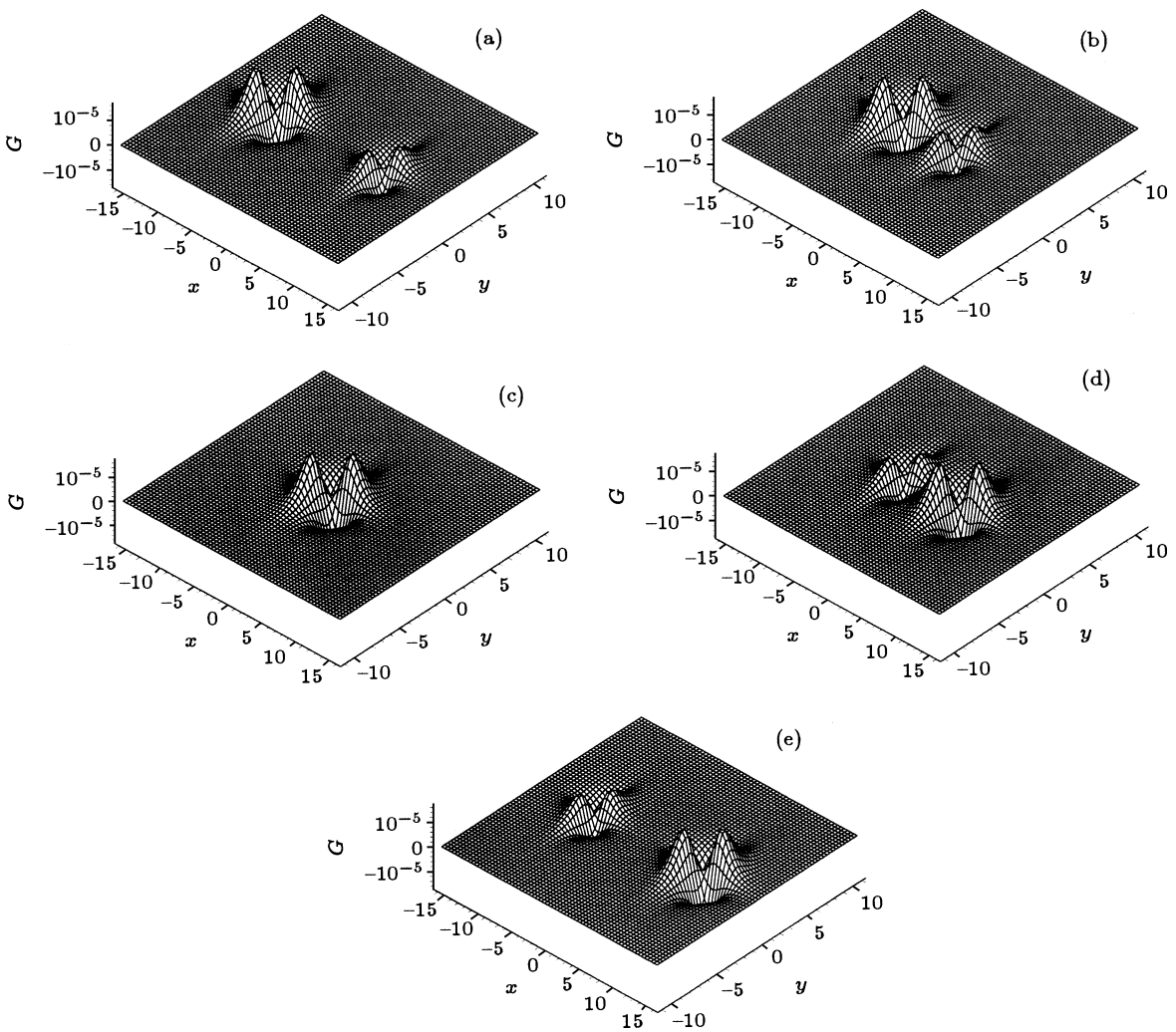



**Fig.1.** The evolutional profile of two solitonic structures for the solution $G$ expressed by Eq.(31) under the condition (32) at different times (a) $t = -8$, (b) $t = -4$, (c) $t = 0$, (d) $t = 4$, (e) $t = 8$, respectively.

While for some specific cases, the interactions between solitons are nonelastic. For example, if $\chi(x,t)$ and $\varphi(y)$ are chosen to be

$$\chi = 1 + 0.1\tanh(x+t) + 0.05\tanh(x-t),$$
$$\varphi = 1 + 0.1\tanh(y), \tag{33}$$

and $\sigma = -1$ in Eq.(31), we can obtain another type of solitary wave solution of Eq.(1) with nonelastic behaviour. Figure 2 shows an evolutional plot of two-dromion solution $G$ expressed by Eq.(31). From Fig.2, we can see that the two solitary wave shapes and their amplitudes are changed after their collisions.

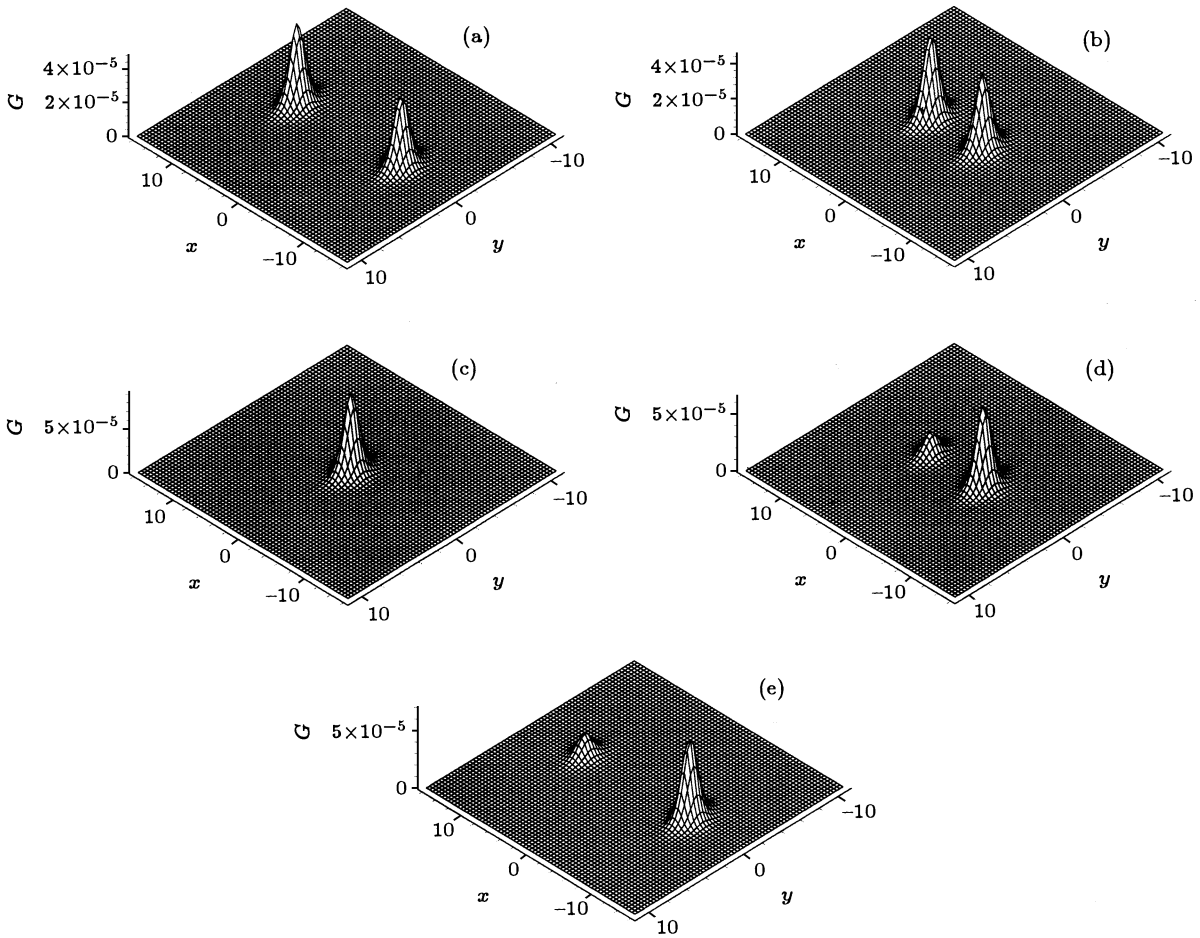



**Fig.2.** The evolutional plot of two-dromion solution $G$ expressed by Eq.(31) under the condition (33) at different times (a) $t = -8$, (b) $t = -4$, (c) $t = 0$, (d) $t = 4$, (e) $t = 8$.

From the above discussions, one can deduce that the solitary wave excitations may possess some novel properties that have not been revealed. Recently, it is reported both theoretically and experimentally that fission and fusion phenomena can happen for solitons or solitary waves, which are essentially equivalent to the completely nonelastic behaviour.[8] Now we focus our attention on these intriguing fusion and fission phenomena for the solitary wave solutions $G$, which may exist under certain circumstances. For instance, when we select the arbitrary functions $\chi$ and $\varphi$ to be

$$\begin{aligned}\chi =&(1+\exp(2x+3t))^2\\&+0.8\exp(2x-3t)(1+\exp(2x-3t))^{-2},\\\varphi =&\exp(2y+3),\end{aligned}\tag{34}$$

and substitute Eq.(34) with $\sigma = -1$ into Eq.(31), we can obtain a new kind of solitary solution of Eq.(1). Figure 3 shows an evolutional profile corresponding to the physical quantity $G$ of the two-dromion solution expressed by Eq.(31), exhibiting a fusion phenomenon for the two solitons. From Fig.3, we can clearly see that the two single-solitons fuse to one soliton finally. Along with this line, considering $\chi$ and $\varphi$ to be

$$\begin{aligned}\chi =&[\exp(5x+5t)+0.8\exp(2x+3t)\\&+0.5\exp(2x+4t)](1+\exp(2x+3t))^{-2},\\\varphi =&\exp(2y+3),\end{aligned}\tag{35}$$

and $\sigma = -1$ for the solution $G$ expressed by Eq.(31), we can obtain another new type of solitary excitation with apparently different properties presented in Fig.4, compared with the fusion phenomenon in Fig.3. From Fig.4, we can find that one single-soliton fissions into three solitons.

Certainly, if we consider other selections for the arbitrary functions $\chi$ and $\varphi$ like Jacobian functions, then we may derive some new solutions with double periodic properties, which are neglected in this paper.

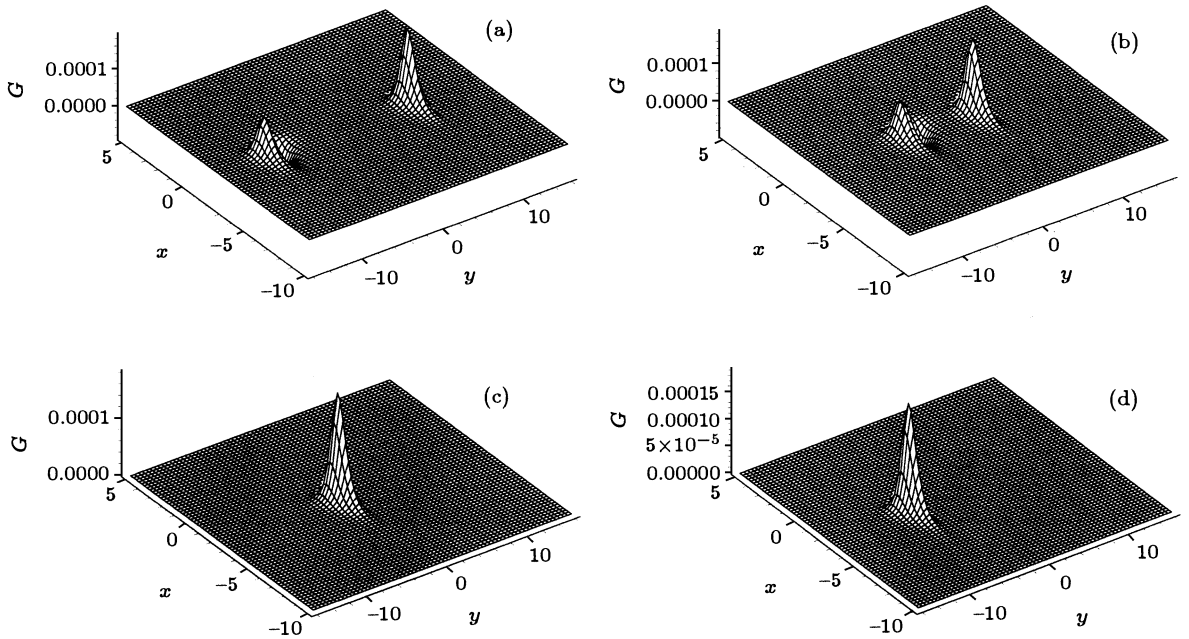



**Fig.3.** The evolutional profile of two single-soliton fusion into one soliton for the solution $G$ expressed by Eq.(31) with condition (34) at different times (a) $t = -6$, (b) $t = -3$, (c) $t = 0$, (d) $t = 3$.

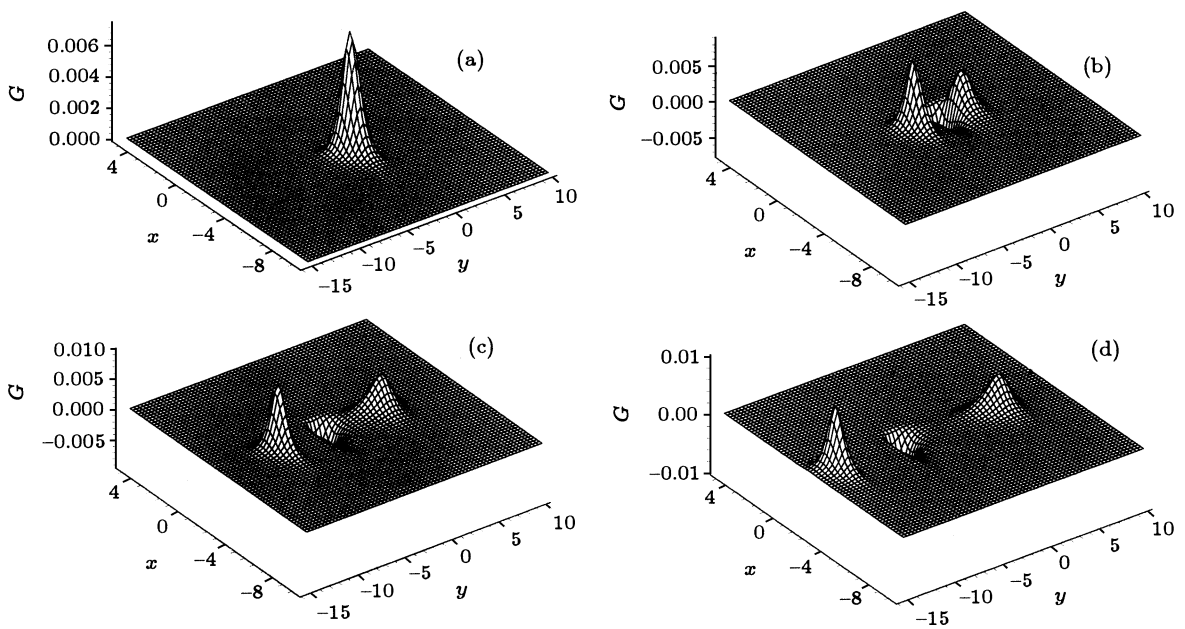



**Fig.4.** One single soliton fission into three solitons evolutional plot of the solution $G$ expressed by (31) with condition (35) at different times (a) $t = 0$, (b) $t = 2$, (c) $t = 4$,(d)$t = 6$, respectively.

# 4. Summary and discussion

In summary, by means of an extended mapping approach, the (2+1)-dimensional BKK system is successfully solved. Based on the derived new type of variable-separable solutions with two arbitrary functions, we have given some specific examples such as soliton fission and fusion solutions of a higher-dimensional BKK system. Usually, it is considered that the interactions among solitons are completely elastic. However, for some specific cases, the soliton interactions may be nonelastic or completely nonelastic. From the above analysis, we can see that these intriguing phenomena may occur in a higher-dimensional soliton system if we choose appropriate initial conditions or boundary conditions for the given system, which are similar to those in some (1+1)-dimensional cases studied by several authors.[6−8] Al-

though we have given out some soliton fusion and fission phenomena in the (2+1)-dimensional case, it is obvious that there are still many significant and interesting problems to be further discussed. As the authors[6−8] have pointed out in (1+1)-dimensional cases, these problems are as follows. What is the necessary and sufficient condition for soliton fission and fusion? What is for soliton elastic and nonelastic interactions? What is the general equation for the distribution of the energy and momentum after soliton fission and soliton fusion? How can we use the soliton fission and soliton fusion of integrable models to investigate the observed soliton fission and soliton fusion in the experiments? These are all the pending issues to be further studied.

Meanwhile, it is interesting to mention whether, based on the solution (30), we can obtain abundant localized excitations such as dromion, lump, peakon, compacton and foldon etc, which are neglected in our present paper. Comparing the general formula (1.1) in Ref.[1] with the solution (30), one finds easily that they are essentially equivalent to each other when choosing $p(x,t) = \chi(x,t)$, $q(y,t) = a_0 + \varphi(y)$, $a_1 = a_2 = 1$ and $a_3 = 0$. Therefore, all the localized excitations based on the general formula (1.1) can be re-derived from the solution (30).

This paper is just our preliminary work. We hope the approach applied in the present work may be further extended to other nonlinear physical systems.

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