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Exact invariants and adiabatic invariants of holonomic system in terms of quasi-coordinates*

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Based on the theory of Lie symmetries and conserved quantities, the exact invariants and adiabatic invariants of holonomic system are studied in terms of quasi-coordinates. The perturbation to symmetries for the holonomic system in terms of quasi-coordinates under small excitation is discussed. The concept of high-order adiabatic invariant is presented, and the form of exact invariants and adiabatic invariants as well as the conditions for their existence are given. Then the corresponding inverse problem is studied.

Keywords: holonomic system in terms of quasi-coordinates, perturbation, exact invariant, adiabatic invariant

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1. Introduction

Introducing the concept of quasi-coordinate is of great significance in mechanical systems, especially in nonholonomic mechanical systems, and is closely related to quasi-velocity.[1−3] The quasi-coordinate is superior to the generalized coordinate, and quasi-velocity is more general than the generalized velocity.

People have paid increasing attention to the study of symmetries and conserved quantities of mechanical systems and many results have sprung up in recent years.[4−20] These results bring our knowledge into the essence of mechanical systems. In 1917, Burgers first proposed the adiabatic invariants which pointed to a special type of Hamilton system.[21] Adiabatic invariants mean that they are almost unchanged when the parameter varies very slowly, and they play a very important role in the research into quasi-integrability of a mechanical system. In fact, the parameter varying very slowly is equivalent to the action of a small perturbation, so we conduct our study by the latter model in this paper.

The study of adiabatic invariants has become a popular subject in mechanics,[22−25] atomic and molecular physics,[26] etc. In this paper, we discuss the exact invariants and adiabatic invariants of holonomic systems in terms of quasi-coordinates from the point of view of Lie symmetries and conserved quantities. First, we construct the equations for a holonomic system in terms of quasi-coordinates, and describe the concept of higher-order adiabatic invariants. Second, we prove the conditions for the existence of the exact invariants and adiabatic invariants, and give their forms. Third, we study the inverse problems of the perturbation to symmetries of the system. Finally, we present an example to illustrate these results.

2. Dynamical equations of motion for the holonomic system in terms of quasi-coordinates

Suppose that the configuration of a holonomic system is determined by n generalized coordinates $q_s(s = 1, \cdots, n)$. Choose n independent and compatible quasi-velocities

$$\omega_s = a_{sk}(\boldsymbol{q})\dot{q}_k \quad (s = 1, \cdots, n), \tag{1}$$

and their reverse transformations

$$\dot{q}_s = b_{sk}(\boldsymbol{q})\omega_k \quad (s = 1, \cdots, n). \tag{2}$$

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Then the dynamical equations of motion for the system can be written in the form[2,3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial \omega_s}-\frac{\partial T^*}{\partial \pi_s}+\frac{\partial T^*}{\partial \omega_k}\gamma_{rs}^k\omega_r=P_s^* \quad (s=1,\cdots,n), \tag{3}$$

where π_s are quasi-coordinates and

$$\gamma_{rs}^k=\left(\frac{\partial a_{kt}}{\partial q_m}-\frac{\partial a_{km}}{\partial q_t}\right)b_{mr}b_{ts},$$
$$T^*(t,q_s,\omega_s)=T(t,q_s,b_{sk}\omega_k), \tag{4}$$

$$\frac{\partial}{\partial \pi_s}=b_{ks}\frac{\partial}{\partial q_k}, \quad P_s^*=Q_k\frac{\partial \dot{q}_k}{\partial \omega_s}. \tag{5}$$

P_s^* is divided into the potential part $P_s^{*\prime}$ and the non-potential part $P_s^{*\prime\prime}$, then Eqs.(3) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s}-\frac{\partial L^*}{\partial \pi_s}+\frac{\partial L^*}{\partial \omega_k}\gamma_{rs}^k\omega_r=P_s^{*\prime\prime} \quad (s=1,\cdots,n). \tag{6}$$

Expanding the above equations, we can derive all $\dot{\omega}_s$ as follows:

$$\dot{\omega}_s=g_s(t,\boldsymbol{q},\boldsymbol{\omega}) \quad (s=1,\cdots,n). \tag{7}$$

3. Lie symmetries and exact invariants

Introducing the infinitesimal transformations of groups of time and quasi-coordinates

$$t^*=t+\Delta t,\pi_s^*(t^*)=\pi_s(t)+\Delta\pi_s, \quad (s=1,\cdots,n), \tag{8}$$

and their expansions

$$t^*=t+\varepsilon\tau^0(t,\boldsymbol{q},\boldsymbol{\omega}), \quad \pi_s^*(t^*)=\pi_s(t)+\varepsilon\xi_s^0(t,\boldsymbol{q},\boldsymbol{\omega}),$$
$$(s=1,\cdots,n), \tag{9}$$

where ε is an infinitesimal parameter, τ^0 and ξ_s^0 are infinitesimal generators. Introduce an infinitesimal generator vector

$$X^{(0)}=\tau^0\frac{\partial}{\partial t}+\xi_s^0\frac{\partial}{\partial \pi_s}, \tag{10}$$

and its first extension

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s^0-\omega_s\dot{\tau}^0)\frac{\partial}{\partial \omega_s}, \tag{11}$$

and its second extension

$$X^{(2)}=X^{(1)}+[(\dot{\xi}_s^0-\omega_s\dot{\tau}^0)^\bullet-\dot{\omega}_s\dot{\tau}^0]\frac{\partial}{\partial \dot{\omega}_s}. \tag{12}$$

The invariance of Eqs.(7) under the infinitesimal transformations (9) leads to the following equations:

$$X^{(2)}(\dot{\omega}_s-g_s(t,\boldsymbol{q},\boldsymbol{\omega}))=0 \quad (s=1,\cdots,n), \tag{13}$$

if

$$\dot{\omega}_s=g_s \quad (s=1,\cdots,n). \tag{14}$$

Substituting Eq.(12) into Eqs.(13) and noticing Eqs.(14), we obtain

$$\ddot{\xi}_s^0-\omega_s\ddot{\tau}^0-2\dot{\tau}^0g_s=X^{(1)}(g_s) \quad (s=1,\cdots,n); \tag{15}$$

Eqs.(15) are called the Lie symmetrical determining equations for a holonomic system in terms of quasi-coordinates.

Definition 1[3] If the generators τ^0 and ξ_s^0 satisfy the determining equations (15), then the corresponding transformations are called the Lie symmetrical transformations, and the corresponding symmetries are called the Lie symmetries.

Proposition 1 For the infinitesimal generators τ^0 and ξ_s^0 satisfying Eq.(15), if there exists a gauge function $G^0=G^0(t,\boldsymbol{q},\boldsymbol{\omega})$ satisfying the following structure equation:

$$L^*\dot{\tau}^0+X^{(1)}(L^*)+K_s(\xi_s^0-\omega_s\tau^0)+\dot{G}^0=0, \tag{16}$$

where

$$K_s=P_s^{*\prime\prime}-\frac{\partial L^*}{\partial \omega_k}\gamma_{rs}^k\omega_r,$$

then the holonomic system in terms of quasi-coordinates has the following exact invariant of Lie symmetries:

$$I_0=L^*\tau^0+\frac{\partial L^*}{\partial \omega_s}(\xi_s^0-\omega_s\tau^0)+G^0=\text{const}. \tag{17}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I_0}{\mathrm{d}t}=&\dot{L}^*\tau^0+L^*\dot{\tau}^0+\frac{\partial L^*}{\partial \omega_s}(\dot{\xi}_s^0-\dot{\omega}_s\tau^0-\omega_s\dot{\tau}^0)\\&+(\xi_s^0-\omega_s\tau^0)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s}\\&-L^*\dot{\tau}^0-X^{(1)}(L^*)-K_s(\xi_s^0-\omega_s\tau^0)\\=&(\xi_s^0-\omega_s\tau^0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s}-\frac{\partial L^*}{\partial \pi_s}-K_s\right)=0.\end{aligned}$$

4. Perturbation to the symmetries and adiabatic invariants

Firstly we give the concept of high-order adiabatic invariants.

Definition 2 If $I_z(t,\boldsymbol{q},\boldsymbol{\omega},\varepsilon)$ is a physical quantity including ε whose highest power is z in a mechanical system, and its derivative with respect to time t is in direct proportion to ε^{z+1}, then I_z is called a zth order adiabatic invariant of the mechanical system.

Suppose the holonomic system in terms of quasi-coordinates corresponding to Eq.(6) is perturbed by

small quantities εW_s, then the equations of motion for the system become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \omega_s}-\frac{\partial L^*}{\partial \pi_s}=K_s+\varepsilon W_s,\quad (s=1,\cdots,n). \tag{18}$$

Due to the action of εW_s, the primary symmetries and invariants of the system may vary. Suppose the variation is a small perturbation based on the symmetrical transformation of the system without perturbation, and $\tau(t,\boldsymbol{q},\boldsymbol{\omega})$ and $\xi_s(t,\boldsymbol{q},\boldsymbol{\omega})$ express the generators of time and space respectively after being perturbed, then

$$\begin{aligned}\tau&=\tau^0+\varepsilon\tau^1+\varepsilon^2\tau^2+\cdots,\\ \xi_s&=\xi_s^0+\varepsilon\xi_s^1+\varepsilon^2\xi_s^2+\cdots,\end{aligned} \tag{19}$$

and they satisfy

$$\begin{aligned}&L^*\dot{\tau}+X^{(1)'}(L^*)+K_s(\xi_s-\omega_s\tau)\\ &+\varepsilon W_s(\xi_s-\omega_s\tau)+\dot{G}=0,\end{aligned} \tag{20}$$

where

$$\begin{aligned}X^{(1)'}=&X^{(0)'}+(\dot{\xi}_s-\omega_s\dot{\tau})\frac{\partial}{\partial\omega_s}\\ =&\tau\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial\pi_s}+(\dot{\xi}_s-\omega_s\dot{\tau})\frac{\partial}{\partial\omega_s}.\end{aligned}$$

G in Eq.(20) is a gauge function. Let

$$G=G^0+\varepsilon G^1+\varepsilon^2G^2+\cdots, \tag{21}$$

$$X^{(1)k}=\tau^k\frac{\partial}{\partial t}+\xi_s^k\frac{\partial}{\partial\pi_s}+(\dot{\xi}_s^k-\omega_s\dot{\tau}^k)\frac{\partial}{\partial\omega_s}. \tag{22}$$

Substituting Eqs.(19) and Eq.(21) into Eq.(20), we obtain

$$\begin{aligned}&L^*\dot{\tau}^k+X^{(1)k}(L^*)+K_s(\xi_s^k-\omega_s\tau^k)\\ &+W_s(\xi_s^{k-1}-\omega_s\tau^{k-1})+\dot{G}^k=0,\\ &(k=0,1,2,\cdots,z);\end{aligned} \tag{23}$$

when $k=0$, the condition $W_s=0$ holds.

Proposition 2 For the holonomic system in terms of quasi-coordinates perturbed by small quantities εW_s, if the generators $\tau^k(t,\boldsymbol{q},\boldsymbol{\omega})$ and $\xi_s^k(t,\boldsymbol{q},\boldsymbol{\omega})$ under infinitesimal transformations satisfy Eq.(23), and $G^k(t,\boldsymbol{q},\boldsymbol{\omega})$ is the gauge function, then

$$\begin{aligned}I_z=&\varepsilon^k[L^*\tau^k+\frac{\partial L^*}{\partial\omega_s}(\xi_s^k-\omega_s\tau^k)+G^k],\\ &(k=0,1,\cdots,z)\end{aligned} \tag{24}$$

is a zth-order adiabatic invariant of the mechanical system.

Proof

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t}=&\varepsilon^k\Big[\dot{L}^*\tau^k+L^*\dot{\tau}^k+\frac{\partial L^*}{\partial\omega_s}(\dot{\xi}_s^k-\dot{\omega}_s\tau^k-\omega_s\dot{\tau}^k)\\ &+(\xi_s^k-\omega_s\tau^k)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial\omega_s}-L^*\dot{\tau}^k-X^{(1)k}(L^*)\\ &-K_s(\xi_s^k-\omega_s\tau^k)-W_s(\xi_s^{k-1}-\omega_s\tau^{k-1})\Big]\\ =&\varepsilon^k\Big[\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial\omega_s}-\frac{\partial L^*}{\partial\pi_s}-K_s\Big)(\xi_s^k-\omega_s\tau^k)\\ &-W_s(\xi_s^{k-1}-\omega_s\tau^{k-1})\Big].\end{aligned}$$

Using Eq.(18), we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^k[\varepsilon W_s(\xi_s^k-\omega_s\tau^k)-W_s(\xi_s^{k-1}-\omega_s\tau^{k-1})].$$

Expanding the above formula and making summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^{z+1}W_s(\xi_s^z-\omega_s\tau^z). \tag{25}$$

This shows that $\mathrm{d}I_z/\mathrm{d}t$ is in direct proportion to ε^{z+1}, so I_z is a zth-order adiabatic invariant of the mechanical system.

5. Inverse problem

Suppose that the system has a first-order adiabatic invariant as follows:

$$I_1=\lambda_0(t,\boldsymbol{q},\boldsymbol{\omega})+\varepsilon\lambda_1(t,\boldsymbol{q},\boldsymbol{\omega}). \tag{26}$$

Differentiating this with respect to t, we have

$$\begin{aligned}\frac{\mathrm{d}I_1}{\mathrm{d}t}=&\frac{\partial\lambda_0}{\partial t}+\Big(\frac{\partial\lambda_0}{\partial\pi_s}\omega_s+\frac{\partial\lambda_0}{\partial\omega_s}\dot{\omega}_s\Big)\\ &+\varepsilon\left[\frac{\partial\lambda_1}{\partial t}+\Big(\frac{\partial\lambda_1}{\partial\pi_s}\omega_s+\frac{\partial\lambda_1}{\partial\omega_s}\dot{\omega}_s\Big)\right].\end{aligned} \tag{27}$$

Multiplying Eq.(18) by $\xi_s-\omega_s\tau$, and taking the summation for s, we have

$$(\xi_s-\omega_s\tau)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial\omega_s}-\frac{\partial L^*}{\partial\pi_s}-K_s-\varepsilon W_s\Big)=0. \tag{28}$$

According to Eqs.(25), and adding Eqs.(27) and (28), we have

$$\begin{aligned}&\frac{\partial\lambda_0}{\partial t}+\Big(\frac{\partial\lambda_0}{\partial\pi_s}\omega_s+\frac{\partial\lambda_0}{\partial\omega_s}\dot{\omega}_s\Big)\\ &+\varepsilon\left[\frac{\partial\lambda_1}{\partial t}+\Big(\frac{\partial\lambda_1}{\partial\pi_s}\omega_s+\frac{\partial\lambda_1}{\partial\omega_s}\dot{\omega}_s\Big)\right]\\ &+(\xi_s-\omega_s\tau)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial\omega_s}-\frac{\partial L^*}{\partial\pi_s}-K_s-\varepsilon W_s\Big)\\ =&\varepsilon^2W_s(\xi_s^1-\omega_s\tau^1),\end{aligned} \tag{29}$$

where the expressions of τ and ξ_s are Eqs.(19).

Firstly, we seek the generators τ^0 and ξ_s^0 of the infinitesimal transformations without perturbation.

Separating the terms not containing ε in Eq.(29), then separating the terms containing $\dot{\omega}_k$ and taking their coefficients as zeros, we have

$$\frac{\partial\lambda_0}{\partial\omega_s}-\frac{\partial^2 L^*}{\partial\omega_s\partial\omega_k}(\xi_k^0-\omega_k\tau^0)=0 \quad (k=1,\cdots,n). \tag{30}$$

Letting $\dfrac{\partial^2 L^*}{\partial\omega_s\partial\omega_k}=H_{sk}$, $h_{sk}=(H_{sk})^{-1}$, here H_{sk} is called the Hess matrix, and from the above equation we obtain

$$\xi_k^0-\omega_k\tau^0=h_{sk}\frac{\partial\lambda_0}{\partial\omega_s}. \tag{31}$$

Further letting

$$\lambda_0=L^*\tau^0+\frac{\partial L^*}{\partial\omega_k}(\xi_k^0-\omega_k\tau^0)+G^0, \tag{32}$$

from Eqs.(31) and (32), we can obtain the items without perturbation:

$$\begin{aligned}\tau^0=&L^{*-1}\left(\lambda_0-\frac{\partial L^*}{\partial\omega_k}h_{sk}\frac{\partial\lambda_0}{\partial\omega_s}-G^0\right),\\ \xi_k^0=&h_{sk}\frac{\partial\lambda_0}{\partial\omega_s}+\omega_k L^{*-1}\left(\lambda_0-\frac{\partial L^*}{\partial\omega_k}h_{sk}\frac{\partial\lambda_0}{\partial\omega_s}-G^0\right).\end{aligned} \tag{33}$$

Furthermore, we can obtain the results of perturbation items of generator functions under small disturbance:

$$\begin{aligned}\tau^1=&L^{*-1}\left(\lambda_1-\frac{\partial L^*}{\partial\omega_k}h_{sk}\frac{\partial\lambda_1}{\partial\omega_s}-G^1\right),\\ \xi_k^1=&h_{sk}\frac{\partial\lambda_1}{\partial\omega_s}+\omega_k L^{*-1}\left(\lambda_1-\frac{\partial L^*}{\partial\omega_k}h_{sk}\frac{\partial\lambda_1}{\partial\omega_s}-G^1\right).\end{aligned} \tag{34}$$

6. An example

Try to study the exact invariants and adiabatic invariants of the Euler dynamical equations of a rigid body rotating around a fixed point, as shown in Fig.1. The intersection line AN of the plane $A\xi\eta$ and the plane AXY is called the nodal line. The angle θ between axis AZ and axis $A\zeta$ is called the nutation angle, the angle ψ between axis AX and axis AN is called the angle of precession, and the angle φ between axis AN and axis $A\xi$ is called the spin angle.

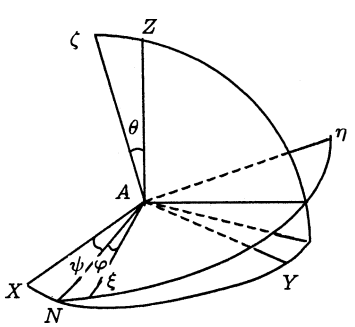


Fig.1. A rigid body rotating about a fixed point.

Firstly, we find the exact invariant. Let

$$q_1=\psi, \quad q_2=\theta, \quad q_3=\varphi; \tag{35}$$

choose quasi-velocities as

$$\begin{aligned}\omega_1&=\dot{q}_1\sin q_2\sin q_3+\dot{q}_2\cos q_3,\\ \omega_2&=\dot{q}_1\sin q_2\cos q_3-\dot{q}_2\sin q_3,\\ \omega_3&=\dot{q}_1\cos q_2+\dot{q}_3.\end{aligned} \tag{36}$$

we have

$$\begin{aligned}\dot{q}_1&=\omega_1\frac{\sin q_3}{\sin q_2}+\omega_2\frac{\cos q_3}{\sin q_2},\\ \dot{q}_2&=\omega_1\cos q_3-\omega_2\sin q_3,\\ \dot{q}_3&=-\omega_1\sin q_3\mathrm{ctg}q_2-\omega_2\cos q_3\mathrm{ctg}q_2+\omega_3;\end{aligned} \tag{37}$$

so we can obtain

$$\begin{aligned}a_{11}&=\sin q_2\sin q_3, \quad a_{12}=\cos q_3, \quad a_{13}=0,\\ a_{21}&=\sin q_2\cos q_3, \quad a_{22}=-\sin q_3, \quad a_{23}=0,\\ a_{31}&=\cos q_2, \quad a_{32}=0, \quad a_{33}=1;\end{aligned} \tag{38}$$

$$\begin{aligned}b_{11}&=\frac{\sin q_3}{\sin q_2}, \quad b_{12}=\frac{\cos q_3}{\sin q_2}, \quad b_{13}=0,\\ b_{21}&=\cos q_3, \quad b_{22}=-\sin q_3, \quad b_{23}=0,\\ b_{31}&=-\sin q_3\mathrm{ctg}q_2, \quad b_{32}=-\cos q_3\mathrm{ctg}q_2, \quad b_{33}=1;\end{aligned} \tag{39}$$

and Boltzmann symbols

$$\begin{aligned}\gamma_{23}^1&=-\gamma_{32}^1=-1,\\ \gamma_{31}^2&=-\gamma_{13}^2=-1,\\ \gamma_{12}^3&=-\gamma_{21}^3=-1,\end{aligned} \tag{40}$$

the others are equal to zero. We have

$$L^*=\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2), \tag{41}$$

where A, B and C are principal moments of inertia. So we can obtain

$$\begin{aligned}&\frac{\partial L^*}{\partial\omega_1}=A\omega_1, \quad \frac{\partial L^*}{\partial\omega_2}=B\omega_2, \quad \frac{\partial L^*}{\partial\omega_3}=C\omega_3,\\ &\frac{\partial L^*}{\partial\pi_1}=\frac{\partial L^*}{\partial\pi_2}=\frac{\partial L^*}{\partial\pi_3}=0,\\ &\frac{\partial L^*}{\partial\omega_k}\gamma_{r1}^k\omega_r=\frac{\partial L^*}{\partial\omega_3}\gamma_{21}^3\omega_2+\frac{\partial L^*}{\partial\omega_2}\gamma_{31}^2\omega_3=(C-B)\omega_2\omega_3,\\ &\frac{\partial L^*}{\partial\omega_k}\gamma_{r2}^k\omega_r=(A-C)\omega_3\omega_1,\\ &\frac{\partial L^*}{\partial\omega_k}\gamma_{r3}^k\omega_r=(B-A)\omega_1\omega_2,\\ &K_1=P_1^*-(C-B)\omega_2\omega_3,\\ &K_2=P_2^*-(A-C)\omega_3\omega_1,\\ &K_3=P_3^*-(B-A)\omega_1\omega_2.\end{aligned}$$

Substituting the above equations into Eq.(6), we obtain the Euler dynamical equations

$$A\dot{\omega}_1+(C-B)\omega_2\omega_3=P_1^*,$$
$$B\dot{\omega}_2+(A-C)\omega_3\omega_1=P_2^*,$$
$$C\dot{\omega}_3+(B-A)\omega_1\omega_2=P_3^*, \quad (42)$$

hence

$$\dot{\omega}_1=\frac{B-C}{A}\omega_2\omega_3+\frac{P_1^*}{A},$$
$$\dot{\omega}_2=\frac{C-A}{B}\omega_3\omega_1+\frac{P_2^*}{B},$$
$$\dot{\omega}_3=\frac{A-B}{C}\omega_1\omega_2+\frac{P_3^*}{C}. \quad (43)$$

Suppose the nonpotential generalized forces are equal to zero, i.e.

$$P_1^*=P_2^*=P_3^*=0,$$

then Eqs.(43) become

$$\dot{\omega}_1=\frac{B-C}{A}\omega_2\omega_3,$$
$$\dot{\omega}_2=\frac{C-A}{B}\omega_3\omega_1,$$
$$\dot{\omega}_3=\frac{A-B}{C}\omega_1\omega_2. \quad (44)$$

The determining equations (15) give

$$\ddot{\xi}_1^0-\omega_1\ddot{\tau}^0-2\dot{\tau}^0\left(\frac{B-C}{A}\omega_2\omega_3\right)$$
$$=\frac{B-C}{A}[\omega_3(\dot{\xi}_2^0-\omega_2\dot{\tau}^0)+\omega_2(\dot{\xi}_3^0-\omega_3\dot{\tau}^0)],$$
$$\ddot{\xi}_2^0-\omega_2\ddot{\tau}^0-2\dot{\tau}^0\left(\frac{C-A}{B}\omega_3\omega_1\right)$$
$$=\frac{C-A}{B}[\omega_1(\dot{\xi}_3^0-\omega_3\dot{\tau}^0)+\omega_3(\dot{\xi}_1^0-\omega_1\dot{\tau}^0)],$$
$$\ddot{\xi}_3^0-\omega_3\ddot{\tau}^0-2\dot{\tau}^0\left(\frac{A-B}{C}\omega_1\omega_2\right)$$
$$=\frac{A-B}{C}[\omega_2(\dot{\xi}_1^0-\omega_1\dot{\tau}^0)+\omega_1(\dot{\xi}_2^0-\omega_2\dot{\tau}^0)]; \quad (45)$$

Eqs.(45) have solutions

$$\tau^0=0,\quad \xi_1^0=\omega_1,\quad \xi_2^0=\omega_2,\quad \xi_3^0=\omega_3. \quad (46)$$

The structure equation (16) gives

$$L^*\dot{\tau}^0+A(\dot{\xi}_1^0-\omega_1\dot{\tau}^0)\omega_1$$
$$+B(\dot{\xi}_2^0-\omega_2\dot{\tau}^0)\omega_2+C(\dot{\xi}_3^0-\omega_3\dot{\tau}^0)\omega_3$$
$$+(B\omega_2\omega_3-C\omega_3\omega_2)(\xi_1^0-\omega_1\tau^0)$$
$$+(-A\omega_1\omega_3+C\omega_3\omega_1)(\xi_2^0-\omega_2\tau^0)$$
$$+(A\omega_1\omega_2-B\omega_2\omega_1)(\xi_3^0-\omega_3\tau^0)+\dot{G}^0=0\,. \quad (47)$$

It has a solution

$$G^0=-\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2). \quad (48)$$

So we can obtain the following exact invariant by Proposition 1:

$$I_0=\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2)=\text{const.} \quad (49)$$

Next we study the adiabatic invariants of the system. Suppose the system is perturbed by the small quantity

$$\varepsilon W_1=-\varepsilon 2A\dot{\omega}_1,\quad \varepsilon W_2=-\varepsilon 2B\dot{\omega}_2,$$
$$\varepsilon W_3=-\varepsilon 2C\dot{\omega}_3. \quad (50)$$

The structure equation (23) gives

$$L^*\dot{\tau}^1+A(\dot{\xi}_1^1-\omega_1\dot{\tau}^1)\omega_1$$
$$+B(\dot{\xi}_2^1-\omega_2\dot{\tau}^1)\omega_2+C(\dot{\xi}_3^1-\omega_3\dot{\tau}^1)\omega_3$$
$$+(B\omega_2\omega_3-C\omega_3\omega_2)(\xi_1^1-\omega_1\tau^1)$$
$$+(-A\omega_1\omega_3+C\omega_3\omega_1)(\xi_2^1-\omega_2\tau^1)$$
$$+(A\omega_1\omega_2-B\omega_2\omega_1)(\xi_3^1-\omega_3\tau^1)$$
$$+A\dot{\omega}_1(\xi_1^0-\omega_1\tau^0)$$
$$+B\dot{\omega}_2(\xi_2^0-\omega_2\tau^0)+C\dot{\omega}_3(\xi_3^0-\omega_3\tau^0)+\dot{G}^1=0. \quad (51)$$

It has solutions as

$$\tau^1=0,\quad \xi_1^1=\omega_1,\quad \xi_2^1=\omega_2,\quad \xi_3^1=\omega_3,$$
$$G^1=\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2). \quad (52)$$

So we can obtain the following first-order adiabatic invariant by Proposition 2:

$$I_1=\frac{1}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2)$$
$$+\varepsilon\frac{3}{2}(A\omega_1^2+B\omega_2^2+C\omega_3^2). \quad (53)$$

Moreover, we can obtain further higher-order adiabatic invariants.

References

[1] Synge J 1936 *Tensorial methods in dynamics* (Toronto, ON: University of Toronto Press)

[2] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing institute of Technology Press) (in Chinese)

[3] Mei F X 1999 *The Application of Lie groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science

Press) (in Chinese)
[4] Vujanovic B D 1995 *Int. J. Non-Linear Mech.* **30** 783
[5] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973
[6] Lakshmanan M and Senthil V M 1992 *J. Phys A: Math. Gen.* **25** 1259
[7] Mei F X, Zhang Y F and Shang M 1999 *Mech. Res. Commun.* **26** 7
[8] Guo Y X 2001 *Chin. Phys.* **10** 181
[9] Zhang R C, Chen X W and Mei F X 2001 *Chin. Phys.* **10** 12
[10] Mei F X 2001 *Chin. Phys.* **10** 177
[11] Qiao Y F 2001 *Acta Phys. Sin.* **50** 811 (in Chinese)
[12] Luo S K 2003 *Chin. Phys.* **12** 841
[13] Luo S K and Cai J L 2003 *Chin. Phys.* **12** 357
[14] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[15] Fu J L and Chen LQ 2004 *Mech. Res. Commun.* **31** 9
[16] Fu J L , Chen L Q and Yang X D 2004 *Chin. Phys.* **13** 287
[17] Zhang Y, Shang M and Mei F X 2000 *Chin. Phys.* **9** 401
[18] Zhang Y and Mei F X 2003 *Chin. Phys.* **12** 1058
[19] Luo S K, Guo Y X and Mei F X 2004 *Acta Phys. Sin.* **53** 2413 (in Chinese)
[20] Fang J H, Chen P S, Zhang J and Li H 2003 *Acta Phys. Sin.* **52** 2945 (in Chinese)
[21] Burgers J M 1917 *Ann. Phys.* **52** 195
[22] Hu J H 1999 *SIAM J. Appl. Math.* **59** 322
[23] Zhao Y Y and Mei F X 1996 *Acta Mech. Sin.* **28** 207 (in Chinese)
[24] Chen X W and Mei F X 2000 *Chin. Phys.* **9** 721
[25] Chen X W and Li Y M 2003 *Chin. Phys.* **12** 936
[26] Ostrovsky V N and Prudov N V 1995 *J. Phys. B: At. Mol. Opt. Phys.* **20** 4435