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The parametric orbits and the form invariance of three-body in one-dimension

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In this paper, the differential equations of motion of a three-body interacting pairwise by inverse cubic forces ("centrifugal potential") in addition to linear forces ("harmonical potential") are expressed in Ermakov formalism in two-dimension polar coordinates, and the Ermakov invariant is obtained. By rescaling of the time variable and the space coordinates, the parametric orbits of the three bodies are expressed in terms of relative energy H_1 and Ermakov invariant. The form invariance of the transformations of two conserved quantities are also studied.

Keywords: three-body, parametric orbits, Ermakov invariant, Hamiltonian function, conserved quantities, form invariance

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1. Introduction

There is a close relation between the symmetry and the conserved quantity in a mechanical system. Research on the conserved quantity and symmetry has already become an important and active field of modern mathematical, mechanical and physical science. The modern methods to find conserved quantities are mainly the Noether method, the Lie method and the form invariance, and a series of important results has been obtained in the past twenty years.[1−10]

Ermakov systems can be represented by the pair of second-order equations, they have attracted attention in the last decades due to their important physical applications and mathematical properties as well.[11−16] A central mathematical property of Ermakov systems is the existence of the Ermakov invariant. Three-body system is a typical system in mechanics and quantum mechanics.[13,17] In this paper, the differential equations of motion of three-body interacting pairwise by inverse cubic forces ("centrifugal potential") in addition to linear forces ("harmonical potential") are expressed in Ermakov formalism in two-dimensional polar coordinates, and the Ermakov invariant is obtained. By rescaling of the time variable and the space coordinates, the parametric orbits of the three-body are expressed in terms of relative energy H_1 and Ermakov invariant. The form invariance of the transformations of two conserved quantities are also studied in this paper.

2. The parametric orbits of the three-body

The Hamiltonian of the three-body in one-dimension is[13]

$$\begin{aligned} H =& \frac{1}{2}(p_1^2+p_2^2+p_3^2) \\ &+ \frac{\sigma^2}{6}[(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2] \\ &+ \frac{g_1}{(x_1-x_2)^2}+\frac{g_2}{(x_2-x_3)^2}+\frac{g_3}{(x_3-x_1)^2}, \end{aligned} \quad (1)$$

where σ, g_1, g_2, and g_3 are non-negative constants. We can set $\sigma = 1$ by a rescaling of time and space coordinates. Moreover, we transform the coordinates to the centre of mass and Jacobi coordinates

$$x_C = \frac{1}{3}(x_1+x_2+x_3), \quad (2)$$

$$x = \frac{1}{\sqrt{2}}(x_1-x_2), \quad (3)$$

$$y = \frac{1}{\sqrt{6}}(x_1+x_2-2x_3). \quad (4)$$

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In these coordinates, the new Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2)^2 + \frac{g_1}{2x^2} + \frac{2g_2}{(x - \sqrt{3}y)^2} + \frac{2g_3}{(x + \sqrt{3}y)^2} + E, \tag{5}$$

where $E = \frac{3}{2}\dot{x}_C^2$ is the energy of the centre of mass.

We next introduce two-dimensional polar coordinates r and θ,

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{x}{y} \tag{6}$$

and

$$x = r\sin\theta, \quad y = r\cos\theta. \tag{7}$$

In these polar coordinates, the Hamiltonian is[17]

$$H = \frac{1}{2}(p_r^2 + \frac{p_\theta^2}{r^2}) + \frac{1}{2}r^2 + \frac{g_1}{2r^2\sin^2\theta} + \frac{g_2}{2r^2\sin^2\left(\theta + \frac{2}{3}\pi\right)} + \frac{g_3}{2r^2\sin^2\left(\theta + \frac{4}{3}\pi\right)} + E. \tag{8}$$

If we choose $g_1 = g_2 = g_3 = g$, together with the trigonometric identity

$$\frac{1}{\sin^2\theta} + \frac{1}{\sin^2\left(\theta + \frac{2}{3}\pi\right)} + \frac{1}{\sin^2\left(\theta + \frac{4}{3}\pi\right)} = \frac{9}{\sin^2 3\theta}, \tag{9}$$

then Eq.(8) turns into

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{1}{2}r^2 + \frac{1}{r^2}\Phi(\theta) + E, \tag{10}$$

in which $\Phi(\theta) = \frac{9g}{2\sin^2 3\theta}$.The Hamiltonian function (8) generates the following equations for the motion of the system:

$$\dot{r} = p_r, \tag{11}$$

$$\dot{\theta} = \frac{p_\theta}{r^2}, \tag{12}$$

$$\dot{p}_r = \frac{p_\theta^2}{r^3} + r + \frac{2\Phi(\theta)}{r^3}, \tag{13}$$

$$\dot{p}_\theta = -\frac{1}{r^2}\frac{\partial\Phi(\theta)}{\partial\theta}. \tag{14}$$

We can easily obtain the second-order equations from the first-order ones:

$$\ddot{r} - r\dot{\theta}^2 + r = \frac{2\Phi(\theta)}{r^3}, \tag{15}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{1}{r^3}\frac{\partial\Phi(\theta)}{\partial\theta}. \tag{16}$$

According to Ref.[16], the Eqs.(15) and (16) are Ermakov systems with $\omega^2 = 1$, $F(\theta) = 2\Phi(\theta)$ and $G(\theta) = \frac{\partial\Phi(\theta)}{\partial\theta}$. The main property of the Ermakov systems is that they possess a constant of motion, the Ermakov invariant,

$$I = \frac{p_\theta^2}{2} + \Phi(\theta). \tag{17}$$

Substituting Eq.(17) into Eq.(10) yields

$$H = \frac{1}{2}p_r^2 + \frac{I}{r^2} + \frac{1}{2}r^2 + E. \tag{18}$$

The canonical Hamiltonian equations can generate the following equations:

$$\dot{r} = \frac{\partial H}{\partial p_r} = \sqrt{2\left(H_1 - \frac{I}{r^2} - \frac{r^2}{2}\right)}, \tag{19}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{r^2}\frac{\partial I}{\partial p_\theta} = \frac{1}{r^2}\sqrt{2(I - \Phi(\theta))}, \tag{20}$$

where $H_1 = H - E$ is the relative energy of the three-body. By rescaling of the time variable and the space coordinates according to

$$q = r^2, \quad \mathrm{d}t = r^2\mathrm{d}\tau, \tag{21}$$

we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = 2q\sqrt{2H_1 q - q^2 - 2I}, \tag{22}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \sqrt{2(I - \Phi(\theta)}. \tag{23}$$

In term of Eqs.(22) and (23), we can obtain the parametric orbits:

$$q = \frac{2I}{H_1 - \sqrt{H_1^2 - 2I}\sin 2\sqrt{2I}(\tau - \tau_1)}, \tag{24}$$

$$\theta = \frac{1}{3}\sin^{-1}\left[\left(1 + \frac{9g}{2I}\right) + \left(1 - \frac{9g}{2I}\right)\sin(6\sqrt{2I}(\tau - \tau_2))\right], \tag{25}$$

where τ_1 and τ_2 are the integration constant arising from Eqs.(22) and (23).These integration constants can always be expressed in terms of the initial position and H_1 and I.

3. The form invariance of the two conserved quantities

The Hamiltonian H is a conserved quantity (energy) because of its time-independence,

$$I_1 = H. \tag{26}$$

Another conserved quantity is the Ermakov invariant in Eq.(17)

$$I_2 = \frac{p_\theta^2}{2} + \Phi(\theta), \tag{27}$$

in which $\Phi(\theta) = \dfrac{9g}{2\sin^2 3\theta}$.

Let us consider infinitesimal transformations in phase space,

$$\begin{aligned} t^* &= t + \varepsilon\tau_\alpha(t,q,p),\\ q_s^* &= q_s + \varepsilon\xi_s^\alpha(t,q,p),\\ p_s^* &= p_s + \varepsilon\eta_s^\alpha(t,q,p),\\ &(s=1,2, \quad \alpha=1,2), \end{aligned} \tag{28}$$

where ε is the infinitesimal parameter, and τ_α, ξ_s^α and η_s^α are the generators of infinitesimal transformations. Under the infinitesimal transformations (28), we introduce the generator vector for infinitesimal transformations

$$X^{(0)} = \tau_\alpha\frac{\partial}{\partial t} + \sum_{s=1}^{2}\xi_s^\alpha\frac{\partial}{\partial q_s} + \sum_{s=1}^{2}\eta_s^\alpha\frac{\partial}{\partial p_s}. \tag{29}$$

According to inverse Noether theorem for Hamilton system, we can obtain the generators of infinitesimal transformations:[1]

$$\xi_s^\alpha = \tau_\alpha\frac{\partial H}{\partial p_s} + \frac{\partial I_\alpha}{\partial p_s}, \tag{30}$$

$$\tau_\alpha = \frac{I_\alpha - \sum\limits_{s=1}^{2}\dfrac{\partial I_\alpha}{\partial p_s}p_s - G_\alpha}{\sum\limits_{s=1}^{2}\dfrac{\partial H}{\partial p_s}p_s - H}, \tag{31}$$

$$\eta_s^\alpha = \frac{\partial p_s}{\partial t}\tau_\alpha + \sum_{k=1}^{2}\frac{\partial p_s}{\partial q_k}\xi_k^\alpha + \sum_{k=1}^{2}\frac{\partial p_s}{\partial\dot{q}_k}(\dot{\xi}_k^\alpha - \dot{q}_k\dot{\tau}_\alpha). \tag{32}$$

Substituting Eq.(26), together with $G_1 = 0$, into Eqs.(30)–(32), we have the following solution:

$$\tau_1 = -1, \quad \xi_1^1 = 0, \quad \xi_2^1 = 0, \quad \eta_1^1 = 0, \quad \eta_2^1 = 0. \tag{33}$$

Substituting Eq. (27) into Eqs.(30)–(32), we obtain

$$\begin{aligned} \tau_2 =& \frac{1}{L}(I_2 - p_\theta^2 - G_2),\\ \xi_1^2 =& \tau_2 p_r,\\ \xi_2^2 =& \frac{\tau_2}{r^2}p_\theta + p_\theta,\\ \eta_1^2 =& \tau_2\dot{p}_r,\\ \eta_2^2 =& r^2\dot{p}_\theta + \tau_2\dot{p}_\theta, \end{aligned} \tag{34}$$

where L is the Lagrangian of the system. Substituting Eqs.(33) and (34), together with Eqs.(15) and (16), into $X^{(0)}(H)$, we find

$$X^{(0)}(H) = 0. \tag{35}$$

Thus

$$\frac{\partial X^{(0)}(H)}{\partial p_r} = 0, \quad -\frac{\partial X^{(0)}(H)}{\partial r} = 0, \tag{36}$$

$$\frac{\partial X^{(0)}(H)}{\partial p_\theta} = 0, -\frac{\partial X^{(0)}(H)}{\partial\theta} = 0. \tag{37}$$

According to Eqs.(36) and (37), we can see that, generators (33) and (34) of the two conserved quantities are form invariance of the system.

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