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# A new non-Noether conserved quantity of the relativistic holonomic nonconservative systems in general Lie transformations*

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For the relativistic holonomic nonconservative system, a new Lie symmetrical non-Noether conserved quantity is given under general infinitesimal transformations of groups in which time is variable. On the basis of the theory of invariance of differential equations of motion under infinitesimal transformations for $t$ and $q_s$, we construct the relativistic Lie symmetrical determining equations and obtain directly a new relativistic Lie symmetrical non-Noether conserved quantity of the system, which only depend on the variables $t$, $q_s$ and $\dot{q}_s$. An example is given to illustrate the application of the results.

**Keywords:** relativity, holonomic nonconservative system, Lie symmetry, non-Noehter conserved quantity

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## 1. Introduction

The conservation laws play an important role in the fields of theoretical physics and analytical mechanics. There are three main symmetry methods used to seek the conserved quantities of a dynamical system: Noether symmetry,[1−4] Lie symmetry[4−7] and Mei's form invariance.[8−10]

In 1979, Lutzky applied the Lie theory to the differential equations of motion for a mechanical system, and studied Lie symmetries and conserved quantities of a dynamical system.[4] In the last 20 years, researchers have developed the method of Lie symmetry and obtained many important results: e.g. Refs.[4–7,11–42]. In general, the Lie symmetry does not imply a conserved quantity. If and only if the infinitesimal transformations of Lie symmetry satisfy the structure equation (i.e. Noether identity), then the Lie symmetry can lead to a conserved quantity,[4−7,11−28] and the conserved quantity is called Noether. In 1992, Hojman[29] gave the direct method of finding a conserved quantity by using Lie symmetry. This direct method is generalized by González-Gascón[30] and Lutzky.[31,32] Pillay and Leach [33] demonstrate that the invariant derived using Hojman's conservation law is trivial for all Noether symmetries. Therefore, using Hojman's method one can find the non-trivial conserved quantity which is led to by the non-Noether symmetry, and the conserved quantity is called non-Noether. In recent years, the Lie symmetrical non-Noether conserved quantities of some dynamical system have been studied by using this direct method in Refs.[34–42] and a series of important results were obtained. But, the direct methods for finding conserved quantity presented above are all only to be utilized under the special infinitesimal transformations of groups in which time is invariant.

Relativistic analytical mechanics is an important aspect for the modern development in the field of theoretical physics. Since 1987, we have constructed the theory of analytical mechanics for the

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relativistic system, and given its basic theoretical frame.[43−48,19,23,24,26] Recently, Refs.[39,40] gave the Hojman conserved quantity of the relativistic Hamiltonian system under special infinitesimal transformations, and Refs.[49,50] gave the non-Noether conserved quantity of the classical Birkhoffian system and the classical Hamiltonian system under general infinitesimal transformations. This paper further studies a new Lie symmetrical non-Noether conserved quantity of a relativistic holonomic nonconservative system under general infinitesimal transformations of groups in which time is variable. On the basis of the theory of invariance of differential equations of motion under infinitesimal transformations, and introducing general infinitesimal transformations for $t$ and $q_s$, we construct the Lie symmetrical determining equations and obtain directly a new Lie symmetrical non-Noether conserved quantity of the system, which only depends on the variables $t$, $q_s$, and $\dot{q}_s$. The corollaries are given in the relativistic Lagrangian system, the special case in which the time is invariant, and the classical holonomic nonconservative system. Also, an example is given to illustrate the application of the results.

## 2. Lie symmetry of the relativistic system

Here we study a relativistic holonomic nonconservative system composed of $N$ particles, and its configuration is determined by the $n$ generalized coordinates $q_s(s = 1, \cdots, n)$. Its motion can be described by the differential equations as follows:[43]

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} = Q_s (s = 1, \cdots, n),\\
&L^*(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = T^* - V, \quad V = V(\boldsymbol{q}),\\
&T^* = \sum_{i=1}^{N} m_{0i} c^2 (1 - \sqrt{1 - \dot{r}_i^2/c^2}),\\
&m_{0i} = m_i \sqrt{1 - \dot{r}_i^2/c^2}, \quad (i = 1, \cdots, N). \qquad (1)
\end{aligned}$$

Here $L^* = L^*(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the relativistic Lagrangian, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the nonpotential generalized forces, and the repeated subscripts represent the summation. In general, it is supposed that the system (1) is nonsingular, i.e.

$$\det\left(\frac{\partial^2 L^*}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0. \qquad (2)$$

From Eq.(1), we can find all generalized accelerations as

$$\ddot{q}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \cdots, n). \qquad (3)$$

We introduce the general infinitesimal transformations for $t$ and $q_s$ as

$$\begin{aligned}
&\bar{t} = t + \varepsilon \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\\
&\bar{q}_s(\bar{t}) = q_s(t) + \varepsilon \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad (4)
\end{aligned}$$

where $\varepsilon$ is an infinitesimal parameter, and $\xi_0$, $\xi_s$ are infinitesimal generators. Equations (4) represent a one-parameter Lie group of transformations. Taking the infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \qquad (5)$$

and its first extended vector

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}, \qquad (6)$$

the invariance of Eq.(3) under the infinitesimal transformations (4) can lead to the satisfaction of the relativistic Lie symmetrical determining equations

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 h_s = X^{(1)}(h_s) \quad (s = 1, \cdots, n). \qquad (7)$$

## 3. A new Lie symmetrical non-Noether conserved quantity of a relativistic system

We now obtain directly the non-Noether conserved quantities from the Lie symmetries of the relativistic holonomic nonconservative system, which only depends on the variables $t$, $q_s$ and $\dot{q}_s$.

**Theorem 1** For the relativistic holonomic nonconservative system (1), if the generators $\xi_0$ and $\xi_s$ of infinitesimal transformations (4) satisfy the determining equation (7) and the following conditions:

$$\ddot{\xi}_0 = 0, \qquad (8)$$

$$\frac{\partial h_s}{\partial \dot{q}_s} = 0, \qquad (9)$$

then the relativistic system possesses the conserved quantity as follows:

$$I_{\mathrm{L}} = \frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_s}{\partial q_s} + \frac{\partial}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) = \text{const.} \qquad (10)$$

**Proof** Using

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial t} = \frac{\partial \dot{\xi}_0}{\partial t} - \frac{\partial h_k}{\partial t}\frac{\partial \xi_0}{\partial \dot{q}_k},\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial \dot{\xi}_s}{\partial q_s} - \frac{\partial h_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k},\\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) = \frac{\partial}{\partial \dot{q}_s}\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)\\
&\quad - \frac{\partial}{\partial q_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) - \frac{\partial h_k}{\partial \dot{q}_s}\frac{\partial}{\partial \dot{q}_k}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \qquad (11)
\end{aligned}$$

we can obtain directly $\mathrm{d}I_{\mathrm{L}} / \mathrm{d}t = 0$.

**Theorem 2** For the relativistic holonomic nonconservative system (1), if the generators $\xi_0$ and $\xi_s$ of infinitesimal transformations (4) satisfy the determining equation (7) and the following condition:

$$\ddot{\xi}_0 = 0, \tag{12}$$

and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial h_s}{\partial \dot{q}_s} + \frac{1}{\mu}\left(\frac{\partial \mu}{\partial t} + \dot{q}_s \frac{\partial \mu}{\partial q_s} + h_s \frac{\partial \mu}{\partial \dot{q}_s}\right) = 0 \quad (s = 1, \cdots, n), \tag{13}$$

then the relativistic system possesses the conserved quantity as follows:

$$I_{\mathrm{L}} = \frac{1}{\mu}\frac{\partial}{\partial t}(\mu \xi_0) + \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}[\mu(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)] = \text{const.} \tag{14}$$

**Proof** We have

$$\begin{aligned}\frac{\mathrm{d}I_{\mathrm{L}}}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial t}\xi_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)\right] \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0).\end{aligned} \tag{15}$$

Substituting Eqs.(11) into Eq.(15) and using Eq.(7), we obtain

$$\begin{aligned}\frac{\mathrm{d}I_{\mathrm{L}}}{\mathrm{d}t} =& \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial t}\xi_0\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)\right] + \ddot{\xi}_0 + \frac{\partial h_s}{\partial \dot{q}_s}\dot{\xi}_0 \\ &+ \frac{\partial^2 h_s}{\partial \dot{q}_s \partial t}\xi_0 + \frac{\partial^2 h_s}{\partial \dot{q}_s \partial q_k}\xi_k \\ &+ \frac{\partial^2 h_s}{\partial \dot{q}_s \partial \dot{q}_k}(\dot{\xi}_k - \dot{q}_k \dot{\xi}_0).\end{aligned} \tag{16}$$

Taking the partial derivatives of condition (13) for $t$, $q_k$ and $\dot{q}_k$, substituting them into Eq.(16), and using condition (12) and Eq.(7), we obtain

$$\frac{\mathrm{d}I_{\mathrm{L}}}{\mathrm{d}t} = 0. \tag{17}$$

## 4. Corollaries

From Theorem 2, we have the following deduction:

**Corollary 1** For the relativistic Lagrangian system (1) satisfying $Q_s \equiv 0$, if the generators $\xi_0$ and $\xi_s$ of infinitesimal transformations (4) satisfy the determining equation (7), and the conditions (12) and (13) hold, then the relativistic system possesses the Lie symmetrical conserved quantity (14).

If we only introduce the special infinitesimal transformations for $q_s$, then we have $\Delta t = 0$ and $\xi_0 = 0$, Eqs.(4)–(7) and (14) become, respectively,

$$\bar{t} = t, \quad \bar{q}_s(\bar{t}) = q_s(t) + \varepsilon \xi_s(t, q, \dot{q}), \tag{18}$$

$$X^{(0)} = \xi_s \frac{\partial}{\partial q_s}, \quad X^{(1)} = X^{(0)} + \dot{\xi}_s \frac{\partial}{\partial \dot{q}_s}, \tag{19}$$

$$\ddot{\xi}_s = X^{(1)}(h_s) = \xi_k \frac{\partial h_s}{\partial q_k} + \dot{\xi}_k \frac{\partial h_s}{\partial \dot{q}_k}, \tag{20}$$

$$I_{\mathrm{L}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}(\mu \dot{\xi}_s) = \text{const.} \tag{21}$$

**Corollary 2** For the relativistic holonomic nonconservative system (1), if the generator $\xi_s$ of infinitesimal transformations (18) satisfies the determining equation (20), and there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying condition (13), then the relativistic system possesses the Lie symmetrical conserved quantity (21).

Under the classical approximation $\dot{\boldsymbol{r}}_i \ll c$,we have $m_i = m_{0i}$. For $T^*$, choosing the former two items of the power-series expression of $(1 - \dot{r}_i^2/c^2)^{1/2}$ with respect to $\dot{r}_i^2/c^2$, we can obtain

$$\begin{aligned}T^* &\approx \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i^2 = T, \\ L^* &\approx T - V = L.\end{aligned} \tag{22}$$

Therefore, from theorem 2, we have the following deduction:

***C*orollary 3** For the classical holonomic nonconservative system (1) satisfy $\dot{\boldsymbol{r}}_i \ll c$, if the generators $\xi_0$ and $\xi_s$ of infinitesimal transformation (4) satisfy the determining equation (7), and the conditions (12) and (13) hold, then the system possesses the Lie symmetrical conserved quantity (14).

For the classical Lagrangian system, if there is a function $\mu = \mu(\boldsymbol{q})$ satisfying condition (13), then we can obtain the Hojman conserved quantity (21).[29]

The above corollaries are also new results.

## 5. Example

As an example, we study the Lie symmetrical non-Noether conserved quantities of a relativistic holonomic nonconservative system, of which the Lagrangian and the nonpotential generalized forces are, respectively,

$$\begin{aligned}L^* &= m_0 c^2 (1 - \sqrt{1 - \dot{q}^2/c^2}) - V, \\ Q &= k_1 (1 - \dot{q}^2/c^2)^{-1/2},\end{aligned} \tag{23}$$

where $V$=constant and $k_1$ = constant. The equation of motion is

$$\ddot{q} = \frac{k_1}{m_0}(1 - \dot{q}^2/c^2) = h. \tag{24}$$

The determining equation (7) gives

$$\ddot{\xi} - \dot{q}\ddot{\xi}_0 - \frac{2k_1}{m_0}(1 - \dot{q}^2/c^2)\dot{\xi}_0 = -\frac{2k_1 q}{m_0 c^2}(\dot{\xi} - \dot{q}\dot{\xi}_0), \tag{25}$$

which has solutions

$$\xi_0 = k_2, \quad \xi = k_3, \tag{26}$$

$$\xi_0 = k_2, \quad \xi = \int \exp\left(-\int \frac{2k_1 q}{m_0 c^2}\mathrm{d}t\right)\mathrm{d}t + k_4 t + k_5, \tag{27}$$

where $k_2$, $k_3$, $k_4$ and $k_5$ are all constant. The generators (26) and (27) respectively satisfy the condition (12). The condition (13) gives

$$-\frac{2k_1\dot{q}}{m_0 c^2} + \frac{\mathrm{d}}{\mathrm{d}t}\ln\mu = 0. \tag{28}$$

And the above equation has following solutions:

$$\mu = \exp\left(\frac{2k_1 q}{m_0 c^2}\right). \tag{29}$$

According to theorem 2, Eqs.(26) and (29) give

$$I_{\mathrm{L}} = \frac{2k_1 k_3}{m_0 c^2} = \text{const.} \tag{30}$$

And Eqs.(27) and (29) give

$$I_{\mathrm{L}} = \frac{2k_1}{m_0 c^2}\left(\int \exp\left(-\int \frac{2k_1 q}{m_0 c^2}\mathrm{d}t\right)\mathrm{d}t + k_4 t + k_5\right) + \frac{\partial}{\partial q}\int \exp\left(-\int \frac{2k_1 q}{m_0 c^2}\mathrm{d}t\right)\mathrm{d}t = \text{const.} \tag{31}$$

Using the inverse Noether theorem, we know that the conserved quantities (30) and (31) are non-Noether.

## References


[1] Noether A E 1918 *Nachr. Akad. Wiss. Math. Phys.* **2** 235
[2] Li Z P 1993 *Classical and Quantal Dynamics of Constrained Systems and Their Symmetry Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)
[3] Li Z P 1999 *Constrained Hamiltonian Systems and Their Symmetry Properties* (Beijing: Beijing Polytechnic University Press) (in Chinese)
[4] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Gonstrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[5] Lutzky M and Math J 1979 *J. Phys.* A **12** 973
[6] Mei F X 2000 *ASME Appl. Mech. Rev.* **53** 283
[7] Mei F X 2000 *Int. J.Non-linear Mech.* **35** 229
[8] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120
[9] Mei F X 2001 *Chin. Phys.* **10** 177
[10] Mei F X 2001 *J. Beijing Inst. Technol.* **10** 138
[11] Prince G E and Elizer C J 1981 *J. Phys. A: Math. Gen.* **14** 587
[12] Thormpson G 1986 *J. Phys. A: Math. Gen.* **19** 105
[13] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380 (in Chinese)
[14] Mei F X, Wu R H and Zhang Y F 1998 *Acta Mech. Sin.* **30** 468
[15] Mei F X, Zhang Y F and Shang M 1999 *Mech. Res. Commun.* **26** 7
[16] Zhang Y and Mei F X 2000 *Chin. Sci. Bull.* **45** 1354
[17] Luo S K 2002 *Chin. Phys. Lett.* **19** 449
[18] Luo S K 2002 *Chin. Phys.* **11** 429
[19] Fu J L and Wang X M 2000 *Acta Phys.Sin.* **49** 1023 (in Chinese)
[20] Zhang Y and Xue Y 2001 *Acta Phys. Sin.* **50** 816 (in Chinese)
[21] Zhang H B and Chen L Q 2002 *Chin. Phys.* **11** 765
[22] Li Y C 2002 *Acta Phys. Sin.* **51** 2186 (in Chinese)
[23] Qiao Y F 2001 *Acta Phys. Sin.* **50** 1(in Chinese)
[24] Fang J H 2002 *Chin. Phys.* **11** 313
[25] Fang J H 2002 *Appl. Math. Mech.* **23** 1215
[26] Fang J H 2003 *Commun. Theor. Phys.* **40** 269
[27] Luo S K 2003 *Acta Phys. Sin.* **52** 2941 (in Chinese)
[28] Luo S K 2004 *Acta Phys. Sin.* **53** 5 (in Chinese)
[29] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** 291
[30] Gonálze-Gascón F 1994 *J. Phys. A: Math. Gen.* **27** 59
[31] Lutzky M 1998 *Int. J. Non-linear Mech.* **33** 393
[32] Lutzky M 1999 *Int. J. Non-linear Mech.* **34** 387
[33] Pillay T and Leach P G L 1996 *J. Phys. A: Math. Gen.* **29** 6999
[34] Mei F X 2002 *Chin. Sci. Bull.* **47** 2049
[35] Mei F X 2003 *Acta Phys. Sin.* **52** 1048 (in Chinese)
[36] Mei F X 2003 *J. Jiangxi Norm. Univ.* **27** 193
[37] Zhang Y 2002 *Chin. Quart. Mech.* **23** 392 (in Chinese)
[38] Zhang Y 2002 *Acta. Phys. Sin.* **51** 461 (in Chinese)
[39] Luo S K 2003 *Chin. Phys. Lett.* **20** 597
[40] Luo S K and JIA L Q 2003 *Commun. Theor. Phys.* **40** 265
[41] Fu J L and Chen L Q 2003 *Phys. Lett.* A **317** 255
[42] Luo S K and Mei F X 2004 *Acta Phys. Sin.* **53** 666 (in Chinese)
[43] Luo S K 1987 *Teach. Math. Commun.* **5** 31
[44] Luo S K 1990 *Proc. ICDVC* (Beijing: Peking University Press) p64
[45] Luo S K 1992*Acta. Math. Sci.* **12** 27 (in Chinese)
[46] Luo S K 1996 *Appl. Math. Mech.* **17** 683
[47] Qiao Y F 2004 *Chin. Phys.* **13** 292
[48] Fang J H 2004 *Commun. Theor. Phys.* **41** 349
[49] Zhang H B and Chen L Q 2004 *Acta Mech. Sin.* **36** 254 (in Chinese)
[50] Fang J H 2004 *Commun. Theor. Phys.* **42** 440