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The scale-transformation of electromagnetic theory and its applications*

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The scale-transformation of electromagnetic theory is investigated in detail based on the form of Maxwell equations in scale-transformation being unchanged in different coordinate systems. The relations of electromagnetic parameters in a rectangular coordinate system and in a spherical coordinate system are presented respectively. The scale-transformation invariants for electromagnetic field are derived and their physical meaning is also presented. It is indicated by simulation that the electromagnetic waves located in medium can be considered to be isotropic due to the fact that the size of propagating vector affected by the scale factors and observing azimuth is on a size of 10^{-9}, which provides a new approach for investigating the electromagnetic characteristics of ellipsoidal targets.

Keywords: scale-transformation, electromagnetic parameter, transformation invariants, electromagnetic scattering

PACC: 0290, 0350D, 4100

1. Introduction

There exist the determined transformation relations among the rectangular coordinate system, spherical coordinate system and cylindrical coordinate system, thereby the Maxwell equations are expressed completely and compactly. The electromagnetic topics related to the targets such as sphere and cylinder are discussed in detail[1−9] and their analytical solutions are presented, which brings much convenience to theoretical study and engineering. The relativity developed by Albert Einstein in 1909 is of epoch-making significance, which provides a limitless vital force for acquiring the solutions of electromagnetic systems moving at a high velocity. It is doubtless that the analytical solutions[12] of scattering field, etc, for the targets of ellipsoids and elliptical cylinders are of great importance in applications. Scale-analyses[11] both for coordinate and for electromagnetic fields offer a novel approach to this kind of problem. In this paper the coordinate scale-analyses are first carried out, and based on the analyses the transformation relations for electric current density vector and electric charge density are studied. It is proved that forms of Maxwell equations are kept unchanged. The relations of the electromagnetic fields, the permittivity tensor ε and permeability tensor μ, the polarization vector and magnetization vector, the propagating vector and their boundary conditions are investigated in detail. The researches on the characteristics of propagating vector in scale-transformation coordinate system are presented finally, which show that the medium space can be considered to be isotropic in the millimetre wave band and the obtained result is compared with the data presented in references.

2. Scale-transformation

2.1. Scale-transformation for an ellipsoidal sphere and coordinate relation

It is assumed that there is an ellipsoidal particle with semi-axes of a, b and b in the system Σ, as shown in Fig.1. The symbols x, y, z and r, θ, φ are respectively used to denote the rectangular coordinates and spherical coordinates in the system Σ. Similarly, x', y', z' and r', θ', φ' are the coordinates in the system

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Σ'. The corresponding coordinate axes in the two systems are parallel with each other.

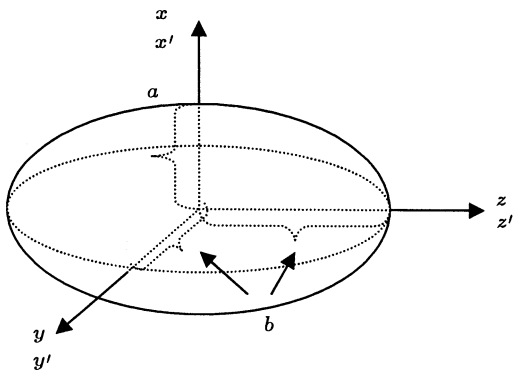


Fig.1. Relations among coordinate axes.

The equation for an ellipsoidal particle in the system Σ is generally written as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{1}$$

In physics, the coordinates and a, b, c in the expression (1) are in units of meter, but the number '1' on the right-hand side is dimensionless. The number "1" can be in units of m^2 after we move the m^2 of the denominators in the terms on the left-hand side to the term on the right-hand side. Thus the symbols a, b, c are dimensionless and called scale factors.

Let

$$x'=\frac{x}{a},\quad y'=\frac{y}{b},\quad z'=\frac{z}{c}, \tag{2a}$$

namely

$$r'=T(a,b,c)\,r, \tag{2b}$$

where

$$T(a,b,c)=\begin{bmatrix} a^{-1} & 0 & 0\\ 0 & b^{-1} & 0\\ 0 & 0 & c^{-1}\end{bmatrix}.$$

Formula (2b) indicates the relation between the two systems. Expression (1) is the equation of a sphere with a radius of one meter. Substituting Eq.(2a) into it yields

$$x'^2+y'^2+z'^2=1. \tag{3}$$

It is concluded from Eq.(3) that an arbitrary ellipsoidal particle is changed into a sphere with a radius of one meter, which, thus, makes it possible to investigate the scattering of electromagnetic wave by an ellipsoidal particle in the way of studying the scattering of electromagnetic wave by a spherical particle. Using formulas (2a) and (3) and relation between rectangular coordinate and spherical coordinate gives the following relations:

$$\begin{aligned} r'\sin\theta'\cos\varphi' &= \frac{r}{a}\sin\theta\cos\varphi,\\ r'\sin\theta'\sin\varphi' &= \frac{r}{b}\sin\theta\sin\varphi,\\ r'\cos\theta' &= \frac{r}{c}\cos\theta. \end{aligned} \tag{4}$$

From the relations (4) we can have the following expressions:

$$\begin{aligned} r' &= r\left(\frac{\sin^2\theta\cos^2\varphi}{a^2}+\frac{\sin^2\theta\sin^2\varphi}{b^2}+\frac{\cos^2\theta}{c^2}\right)^{1/2}\\ &= rq,\\ q &= \left(\frac{\sin^2\theta\cos^2\varphi}{a^2}+\frac{\sin^2\theta\sin^2\varphi}{b^2}+\frac{\cos^2\theta}{c^2}\right)^{1/2}, \end{aligned} \tag{5a}$$

$$\begin{aligned} \sin\theta' &= \frac{\sin\theta g}{q},\quad g=\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)^{1/2},\\ \cos\theta' &= \frac{\cos\theta/c}{q},\\ \sin\varphi' &= \frac{\sin\varphi/b}{g},\quad \cos\varphi'=\frac{\cos\varphi/a}{g}. \end{aligned} \tag{5b}$$

Expression (5a) is the scale relation. It can be seen that the three components in a spherical system are not independent but affected by the scale factors. In some cases such as most raindrops, fog and dust-storms, there exist two factors that are equal to each other, say, $b=c$. Therefore the following are obtained:

$$\begin{aligned} \sin\theta' &= \sin\theta\left(\frac{m^2\cos^2\varphi+\sin^2\varphi}{m^2\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta}\right)^{1/2},\\ \cos\theta' &= \frac{\cos\theta}{\left(m^2\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta\right)^{1/2}},\\ \sin\varphi' &= \frac{\sin\varphi}{\left(m^2\cos^2\varphi+\sin^2\varphi\right)^{1/2}},\\ \cos\varphi' &= \frac{m\cos\varphi}{\left(m^2\cos^2\varphi+\sin^2\varphi\right)^{1/2}},\quad m=\frac{b}{a}. \end{aligned}$$

Expressions (5) are the coordinate relations in a spherical coordinate system and their inverse relations are not derived here. It is noticeable that the rectangular coordinates are independent of each other.

2.2. Relations of differential functors

It is well known that the differential Maxwell equations depict accurately the law of electromagnetic field. In these equations the differential functors play an important role. Due to the fact that in the formula (2a) the same components are dependent on, and the different ones are independent of each other, the following are obtained:

$$\frac{\partial}{\partial x}=\frac{1}{a}\frac{\partial}{\partial x'},\quad \frac{\partial}{\partial y}=\frac{1}{b}\frac{\partial}{\partial y'},\quad \frac{\partial}{\partial z}=\frac{1}{c}\frac{\partial}{\partial z'}.$$

The inverse relations can be easily derived as follows:

$$\frac{\partial}{\partial x'}=a\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y'}=b\frac{\partial}{\partial y},\quad \frac{\partial}{\partial z'}=c\frac{\partial}{\partial z}.$$

For simplicity, we write them in a form of matrix

$$\nabla = T\,(a,b,c)\,\nabla', \tag{6}$$

$$\nabla' = T^{-1}\,(a,b,c)\,\nabla. \tag{7}$$

Expressions (6) and (7) are the relations in a rectangular coordinate system. Their relations in spherical system are easily obtained among the scale factors and not shown here.

2.3. Relations of electromagnetic field

2.3.1. Relations between vectors D and B

It is well known that the electric charge is an invariant in any coordinate system. We thus obtain the following expression:

$$\rho'\,(x',y',z') = abc\rho\,(x,y,z)\,, \tag{8a}$$

which is the scale relation for electric charge density. We now consider the surface-charge density in a plane of x-y, which is denoted with σ_z and σ'_z in the systems Σ and Σ' respectively. They are defined as

$$\sigma'_z=\frac{\mathrm{d}^2q}{\mathrm{d}x'\mathrm{d}y'},\quad \sigma_z=\frac{\mathrm{d}^2q}{\mathrm{d}x\mathrm{d}y}.$$

Because of $\mathrm{d}x'\mathrm{d}y'=\mathrm{d}x\mathrm{d}y/(ab)$, the following expression is therefore obtained:

$$\sigma'_z=ab\sigma_z.$$

With the same method we also acquire

$$\sigma'_x=bc\sigma_x,\quad \sigma'_y=ac\sigma_y.$$

The above can be written in matrix form:

$$\sigma'=M^{-1}\sigma, \tag{8b}$$

where

$$M^{-1}=\begin{bmatrix} bc & 0 & 0\\ 0 & ac & 0\\ 0 & 0 & ab\end{bmatrix}.$$

Equation (8b) is the scale-transformation relation for surface-charge density. The electric-charge conservation law should have an unchanged form in both coordinate systems, namely

$$\nabla'\cdot J'+\frac{\partial\rho'}{\partial t}=0,\quad \nabla\cdot J+\frac{\partial\rho}{\partial t}=0.$$

Utilizing the expressions (6)–(8) and the above law, we obtain the transformation relation about electric current density (ECD)

$$J'_x=bcJ_x,\quad J'_y=acJ_y,\quad J'_z=abJ_z.$$

The above can also be written in a form of matrix as follows

$$J'=\begin{bmatrix} bc & 0 & 0\\ 0 & ac & 0\\ 0 & 0 & ab\end{bmatrix}J=M^{-1}\cdot J. \tag{9a}$$

Formula (9a) is straight derived from the definition of ECD. The line electric current density is given as follows when the current flows in the direction perpendicular to the x-axis:

$$\alpha'_x=\frac{\mathrm{d}I}{\mathrm{d}x'},\quad \alpha_x=\frac{\mathrm{d}I}{\mathrm{d}x}.$$

Since $\mathrm{d}x'=\mathrm{d}x/a$, we have

$$\alpha'_x=a\alpha_x.$$

In the same way we can obtain

$$\alpha'_y=b\alpha_y,\quad \alpha'_z=c\alpha_z.$$

Its matrix form is expressed as

$$\alpha'=T^{-1}\alpha. \tag{9b}$$

It is concluded that the line electric current density has a complete relation as a differential functor.

Since the Maxwell equations have the same form in the systems Σ and Σ', we easily obtain the following formulae:

$$\nabla'\cdot D'=\rho',\quad \nabla\cdot D=\rho,$$
$$\nabla'\cdot B'=0,\quad \nabla\cdot B=0.$$

We acquire the expression (10a) after the relations of electric charge density and differential functor have been inserted into both sides of the above and afterwards compared:

$$D' = M^{-1}D, \quad B' = M^{-1}B. \tag{10a}$$

Expression (10) can keep the forms of Maxwell equations unchanged in the two coordinate systems. The expressions (9a) and (10a) are the transformation in the rectangular coordinate system. It is assumed that the following relation exists between the rectangular coordinate and the spherical coordinate:

$$F_r = PF_s \quad \text{or} \quad F_s = P^{-1}F_r,$$

where P is the transformation matrix.[9] Substituting the above into the expression (9a), we obtain

$$J'_s = P'M^{-1}P^{-1}J_s = AJ_s. \tag{9c}$$

It is also obtained that

$$D'_s = AD_s, \quad B'_s = AB_s, \tag{10b}$$

where

$$\begin{aligned}
A_{11} =& \frac{1}{q}\left[\frac{bc}{a}\sin^2\theta\cos^2\varphi \right.\\
&\left. + \frac{ac}{b}\sin^2\theta\sin^2\varphi + \frac{ab}{c}\cos^2\theta\right],\\
A_{12} =& \frac{\sin 2\theta}{2q}\left[\frac{bc}{a}\cos^2\varphi + \frac{ac}{b}\sin^2\varphi - \frac{ab}{c}\right],\\
A_{13} =& \frac{\sin\theta\sin 2\varphi}{2q}\left[\frac{ac}{b} - \frac{bc}{a}\right],\\
A_{21} =& \frac{\sin 2\theta}{2q}\left[\frac{b}{ag}\cos^2\varphi + \frac{a}{bg}\sin^2\varphi - gab\right],\\
A_{22} =& \frac{1}{q}\left[\frac{b}{ag}\cos^2\varphi\cos^2\theta \right.\\
&\left. + \frac{a}{bg}\cos^2\theta\sin^2\varphi + gab\sin^2\theta\right],\\
A_{23} =& \frac{\sin 2\theta}{2qg}\left[-\frac{b}{a} + \frac{a}{b}\right],\\
A_{31} =& 0,\\
A_{32} =& 0,\\
A_{33} =& \frac{c}{g}.
\end{aligned}$$

The expressions (9c) and (10b) are transformation relations for vectors $\boldsymbol{J}$, $\boldsymbol{D}$ and $\boldsymbol{B}$ in the spherical coordinate system after the scale transforming. The following relations are deduced obviously with all the scale factors being equal to 1:

$$\begin{aligned}
A_{ij} &= 0, \quad i \neq j,\\
A_{ii} &= 1, \quad i = 1, 2, 3.
\end{aligned}$$

The above results imply that the expressions (9c) and (10b) are both correct.

2.3.2. Scale-transformation of electric field and magnetic field

Two curl field equations of Maxwell theory are written as

$$\begin{aligned}
\nabla \times E &= -\frac{\partial B}{\partial t},\\
\nabla \times H &= J + \frac{\partial D}{\partial t}.
\end{aligned}$$

Inserting formulae (6) and (10) into the first of the above yields

$$M^{-1}(T\nabla') \times E = -\frac{\partial B'}{\partial t}.$$

Comparing the above with the expression

$$\nabla' \times E' = -\frac{\partial B'}{\partial t},$$

the following transformation is obtained

$$E' = T^{-1}E. \tag{11a}$$

The relation for magnetic field is also obtained with the same way

$$H' = T^{-1}H. \tag{12}$$

The transformations of (11a) and (12) are acquired in the rectangular coordinate system, but in the spherical system they have the forms

$$E'_{\mathrm{s}} = QE_{\mathrm{s}}, \quad H'_{\mathrm{s}} = QH_{\mathrm{s}}. \tag{11b}$$

Where $Q = P'T^{-1}P^{-1}$. Its elements are expressed as follows

$$\begin{aligned}
Q_{11} =& \frac{1}{q}, \quad Q_{12} = 0, \quad Q_{13} = 0,\\
Q_{21} =& \frac{\sin 2\theta}{2q}\left[\frac{\cos^2\varphi}{cg} + \frac{\sin^2\varphi}{cg} - cg\right],\\
Q_{22} =& \frac{1}{q}\left[\frac{\cos^2\varphi\cos^2\theta}{cg} + \frac{\cos^2\theta\sin^2\varphi}{cg} + cg\sin^2\theta\right],\\
Q_{23} =& 0, \quad Q_{31} = \frac{\sin 2\varphi\sin\theta}{2g}\left[\frac{b}{a} - \frac{a}{b}\right],\\
Q_{32} =& \frac{\sin 2\varphi\cos\theta}{2g}\left[\frac{b}{a} - \frac{a}{b}\right],\\
Q_{33} =& \frac{1}{g}\left[\frac{b}{a}\cos^2\varphi + \frac{a}{b}\sin^2\varphi\right].
\end{aligned}$$

It can be seen that the elements on the diagonal are all zero as the scale factors equal each other, which is

presupposed. Thus the Maxwell equations in the system Σ' are obtained by utilizing the expressions (10a), (11a) and (12a) and Maxwell equations in the system Σ':

$$\begin{aligned}
&\nabla' \cdot D' = \rho', \\
&\nabla' \cdot B' = 0, \\
&\nabla' \times E' = -\frac{\partial B'}{\partial t}, \\
&\nabla' \times H' = J' + \frac{\partial D'}{\partial t}.
\end{aligned} \tag{13}$$

Obviously, the forms of the equations remain the same as those in the system Σ and are irrelative to frequency. Therefore the forms of potential introduced in a general system such as A, Ψ, Hertz vector Π, Debye potential PQ, etc, hold in the system Σ', since the boundary conditions for electromagnetic field, which will be given later, are the same as those in a conventional system. It is easily verified that

$$E' \cdot r' = E \cdot r$$

is a scale flexing invariant just as the Lorentz invariant in relativity, which has the physical meaning that the electric potential between two arbitrary points is a constant.

2.3.3. Transformation of polarization vector P and magnetization vector M

It is well known that the polarization vector for medium is defined as

$$P = \lim_{\Delta v \to 0} \frac{\Sigma(q_i r_i)}{\Delta V}.$$

Inserting the expression (2a) into the above yields

$$P = MP'. \tag{14a}$$

It is concluded that the transformation for $\boldsymbol{P}$ in the spherical system, denoted with the subscript s, is completely like that for an electric displacement vector for $\boldsymbol{P}$ and $\boldsymbol{D}$ have the same transforming relation. Thus the following is derived:

$$P'_{\mathrm{s}} = AP_{\mathrm{s}}. \tag{14b}$$

The transformation for magnetization vector $\boldsymbol{M}_0$ denoted with the subscript 0 to discriminate it from the matrix M, is also obtained as follows:

$$M_0 = TM'_0, \tag{15a}$$

$$M'_{0\mathrm{s}} = QM_{0\mathrm{s}}. \tag{15b}$$

Thus the transformations for $\boldsymbol{P}$ and $\boldsymbol{M}_0$ are entirely obtained.

2.3.4. Transformation for ε_0 and μ_0

Inserting the expressions (10), (11), (12), (14) and (15) into the following expressions:

$$D = \varepsilon_0 E + P, \quad B = \mu_0 (H + M).$$

and comparing the above with

$$D' = \varepsilon'_0 E' + P', \quad B' = \mu'_0 (H' + M').$$

The following are obtained:

$$\begin{aligned}
&\varepsilon'_0 = \varepsilon_0 M^{-1} T, \\
&\mu'_0 = \mu_0 M^{-1} T.
\end{aligned} \tag{16a}$$

It is easy to obtain the relation in the spherical coordinate system for the electromagnetic parameters as follows:

$$\varepsilon'_{0\mathrm{s}} = \varepsilon_0 AQ^{-1}, \quad \mu'_{0\mathrm{s}} = \mu_0 AQ^{-1}. \tag{16b}$$

It is notable that the free space now is anisotropic after scale transforming. We conclude from the above derivation that the expressions (16) hold true in medium. Inserting the expressions (10), (11) and (12) into the following relations

$$B = \mu H, \quad D = \varepsilon E,$$

we acquire

$$\varepsilon'_{\mathrm{s}} = \varepsilon AQ^{-1}, \quad \mu'_{\mathrm{s}} = \mu AQ^{-1}. \tag{16c}$$

Obviously, the free space and medium space become anisotropic and their parameters are all diagonal matrices of 3×3 in the rectangular coordinate system.

2.3.5. The relation of propagating vector

The previous derivations show that the electromagnetic wave existing in the space is no longer isotropic, so the propagating vector must be different from that in a general system. It is supposed that the electromagnetic field and its sources have the form of $\mathrm{e}^{\mathrm{j}(k\cdot r-\omega t)}$. Applying these suppositions to Maxwell equations, we obtain

$$\begin{aligned}
&k \times E = \omega B, \\
&k \cdot B = 0, \\
&k \times H = J - \omega D, \\
&k \cdot D = \rho.
\end{aligned}$$

Inserting the inverse formula derived from the expressions (10) and (11) into the first in the above, the following is obtained:

$$(ak_x \boldsymbol{i} + bk_y \boldsymbol{j} + ck_z \boldsymbol{k}) \times E' = \omega B'.$$

Therefore we obtain

$$k' = T^{-1}k. \tag{17a}$$

This is just as what we anticipated in the beginning of this section. The relation in the spherical coordinate system is directly presented here

$$k'_{\rm s} = Qk_{\rm s}. \tag{17b}$$

The relation (17a) is universal one since it can be also obtained from the others of Maxwell equations, which ensures the consistency of the electromagnetic theory. The following is easily derived from the relations (2a) and (17a):

$$k' \cdot r' = k \cdot r. \tag{17c}$$

It is seen that the propagating phase is a scale- transforming invariant. We further prove that

$$E' \cdot D' \mathrm{d}v', \quad H' \cdot B' \mathrm{d}v',$$

and their summation are also scale-transforming invariants. The derivation is presented in the following:

$$\begin{aligned} E' \cdot D' \mathrm{d}v' =& E' D'^{t} \mathrm{d}v' \\ =& T^{-1} E \left(M^{-1} D\right)^{t} \mathrm{d}v/(abc), \\ T^{-1} E \left(M^{-1} D\right)^{t} \mathrm{d}v/(abc) =& \frac{T^{-1}M^{-1}}{abc} E \cdot D \mathrm{d}v \\ =& E \cdot D \mathrm{d}v. \end{aligned}$$

Utilizing the same method $B \cdot H dv$ is testified to be a scale-transformation invariant. Their sum is therefore an invariant and the physical meaning of the sum is obvious due to the fact that energy, electric charge and electric potential, etc, are external, which are independent of observers' volitions.

3. Investigation of boundary condition

The boundary conditions of electromagnetic field are of great importance in theoretical study and engineering. Problems relative to electromagnetic field are unable to be investigated if there exist no boundary conditions. Among many boundary conditions, three kinds of conditions are often used: namely conditions of electromagnetic field, conditions of electric current density & electric charge density and conditions of propagating vector. These conditions are investigated in the following briefly, which is a basis of solving the problems about electromagnetic wave propagating in the scale-transformation system.

Let the normal of an element surface be in the direction of x-axis. The electric displacement vector can be written in the Σ system as

$$D_{2x} - D_{1x} = \sigma_x.$$

Multiplying the both sides of the above by the scale factor bc and using the formula (8b), the following expression is obtained:

$$D'_{2x} - D'_{1x} = \sigma'_x.$$

It is concluded that the term for $\boldsymbol{D}$ does not vary with scale -transformation. Terms for $D_{y,z}$, $B_{x,y,z}$ are no longer testified here because of their sameness in the system Σ'. Suppose that the normal of an element surface is in the direction of z-axis, thus the tangential component of magnetic field will direct to the y-direction and line electric current density will be in the x-direction, as shown in Fig.2.

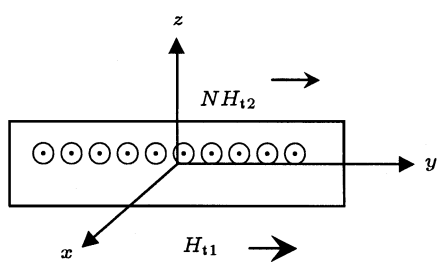


Fig.2. Boundary relation of magnetic field.

The boundary condition in the system Σ is written as

$$H_{t2} - H_{t1} = \alpha.$$

Multiplying both sides of the above by the scale factor b and combing the expression (9b), the boundary condition in Σ' system is acquired as follows:

$$H'_{t2} - H'_{t1} = \alpha'.$$

The boundary condition for electric field in system Σ' is easily derived by the same process and there is no need to deal with it here in detail. In fact, the above conditions are anticipated since the Maxwell theory for electromagnetic field in the system Σ' is identical with that in a general system. Thus we can deduce that the condition of propagating vector also varies with scale transformation. We have proved that boundary conditions are the same in both coordinate systems, though the specific direction is adopted in the above discussion for simplicity. The conclusions should be universal due to the fact that an arbitrary direction of an element surface can be changed into the specific direction via the rotation of coordinate axes.

4. Electromagnetic waves in scale transforming system (STS)

In the scale-transformation system $\sum'$, the electromagnetic wave existing in the space is anisotropic, namely

$$D' = \varepsilon' \cdot E', \quad B' = \mu' \cdot H'.$$

Therefore

$$E' = \varepsilon'^{-1} \cdot D', \quad H' = \mu'^{-1} \cdot B',$$

where

$$\varepsilon'^{-1} = \frac{1}{\varepsilon_0} \begin{bmatrix} \frac{a}{bc} & 0 & 0 \\ 0 & \frac{b}{ac} & 0 \\ 0 & 0 & \frac{c}{ab} \end{bmatrix},$$

$$\mu'^{-1} = \frac{1}{\mu_0} \begin{bmatrix} \frac{a}{bc} & 0 & 0 \\ 0 & \frac{b}{ac} & 0 \\ 0 & 0 & \frac{c}{ab} \end{bmatrix}. \tag{18}$$

In the STS, the time harmonic Maxwell equations are written as

$$\begin{aligned} &k' \times E' = \omega B', \\ &k' \cdot B' = 0, \\ &k' \times H' = -\omega D', \\ &k' \cdot D' = 0. \end{aligned} \tag{19}$$

Obviously, the vectors E' and H' are not perpendicular to the propagating vector K', but the vectors K', D', B' are perpendicular to each other and obey the right-hand rule. The relation between STS (or x-y-z system) and $\boldsymbol{K'}$-$\boldsymbol{D'}$-$\boldsymbol{B'}$ system is shown in Fig.3

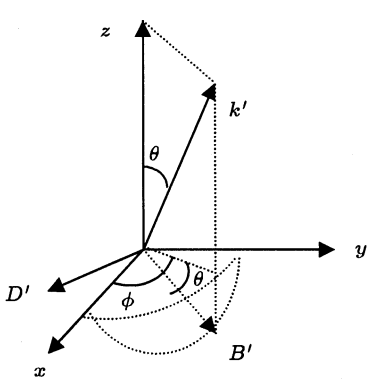


Fig.3. Relation between x-y-z and k-D-B.

The transforming matrix for any vector from the coordinate system x-y-z to the coordinate system k-D-B is given by

$$T = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{bmatrix}. \tag{20}$$

We can easily prove that $T^t = T^{-1}$. The relations between the vectors in the two coordinate systems are obtained as follows:

$$\begin{aligned} &E = T^{-1} \cdot E', \quad D = T^{-1} \cdot D', \\ &H = T^{-1} \cdot H', \quad B = T^{-1} \cdot B'. \end{aligned} \tag{21}$$

The parameters for the medium in the K-D-B system are derived from the expressions (18), (20) and (21) as follows

$$\varepsilon_{kDB}^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix},$$

$$\mu_{kDB}^{-1} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} k_{11} =& \frac{a}{bc\varepsilon_0}\sin^2\phi + \frac{b}{ac\varepsilon_0}\cos^2\phi, \\ k_{12} =& k_{21} = \frac{\frac{a}{bc} - \frac{b}{ab}}{2\varepsilon_0}\cos\theta\sin 2\phi, \\ k_{13} =& k_{31} = \frac{\frac{a}{bc} - \frac{b}{ab}}{2\varepsilon_0}\sin\theta\sin 2\phi, \\ k_{22} =& \frac{1}{\varepsilon_0}\left(\frac{a}{bc}\cos^2\phi + \frac{b}{ab}\sin^2\phi\right)\cos^2\theta \\ &+ \frac{c}{ab\varepsilon_0}\sin^2\theta, \\ k_{23} =& k_{32} = \frac{\sin 2\phi}{2\varepsilon_0}\left(\frac{a}{bc}\cos^2\phi - \frac{b}{ac}\sin^2\phi - \frac{c}{ab}\right), \\ k_{33} =& \frac{1}{\varepsilon_0}\left(\frac{a}{bc}\cos^2\phi + \frac{b}{ac}\sin^2\phi\right)\sin^2\theta \\ &+ \frac{c}{ab\varepsilon_0}\cos^2\theta. \end{aligned}$$

The first expression in the expressions (22) is the result for permittivity in the k–D–B system. It is also concluded from the expression (18) that $\boldsymbol{\varepsilon}_0$ and $\boldsymbol{\mu}_0$ are symmetrical in the scale-transforming system, and the expression for $\boldsymbol{\mu}_0$ is hence acquired by changing ε_0 into μ_0 in the expression (22). The following conclusion is obtained from Eq.(22):

$$k_{11} + k_{22} + k_{33} = \frac{1}{\varepsilon_0}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right).$$

The above expression is the trace which is identical with the trace in a general system as the scale factors are equal to each other, namely $a = b = c$=1. Now the Maxwell equations become

$$\begin{aligned} &k' \times \left(\varepsilon_{kDB}^{-1} \cdot D'\right) = \omega B', \\ &k' \cdot B' = 0, \\ &k' \times \left(\mu_{kDB}^{-1} \cdot B'\right) = -\omega D', \\ &k' \cdot D' = 0. \end{aligned} \tag{23}$$

Since $\boldsymbol{k}' = k' e_3$ in the k-D-B system, the following expression is determined:

$$k' \times A = k \cdot A,$$

where

$$k = \begin{bmatrix} 0 & -k' & 0 \\ k' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first and the third in the expression (23) are now changed into

$$\begin{bmatrix} 0 & -k' & 0 \\ k' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} D_1' \\ D_2' \\ 0 \end{bmatrix} = \omega \begin{bmatrix} B_1' \\ B_2' \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -k' & 0 \\ k' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} B_1' \\ B_2' \\ 0 \end{bmatrix} = -\omega \begin{bmatrix} D_1' \\ D_2' \\ 0 \end{bmatrix}.$$

Combining the expression (22) and the above yields

$$\begin{aligned} k' &= \frac{k_0}{\left(\dfrac{\sin^2\theta\cos^2\phi}{a^2} + \dfrac{\sin^2\theta\sin^2\phi}{b^2} + \dfrac{\cos^2\theta}{c^2}\right)^{1/2}}, \\ \frac{D_2'}{D_1'} &= \frac{0}{1 - \dfrac{k'^2}{k_0^2}\left(\dfrac{\sin^2\theta\cos^2\phi}{a^2} + \dfrac{\sin^2\theta\sin^2\phi}{b^2} + \dfrac{\cos^2\theta}{c^2}\right)}. \end{aligned} \tag{24}$$

Combining the two expressions, we can obtain

$$D_2' \times 0 = D_1' \times 0. \tag{25}$$

Formula (25) implies that the amplitudes of electromagnetic wave are not confined in the directions of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. The propagating vector varies with observing direction, but has a certain magnitude. The conclusion that $k' = k_0$ is what we expected beforehand with the scale factors being $a = b = c$=1. There is no double refraction since there is one velocity in one direction, which implies that the free space is changed into a special one after the scale transforming. The propagating vector varying with the observing points in the K-D-B system is shown in Fig.4.

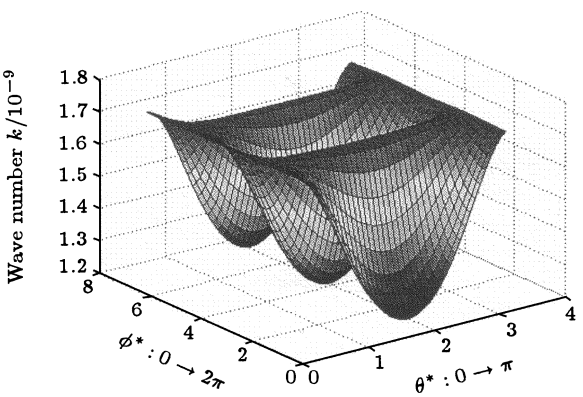


Fig.4. Size of wavenumber varies with observing points. The semi-axes are a=0.01 and b=c=0.014.

It is concluded from the above figure that the wavenumber varies with the azimuth, but the difference between the maximum and minimum is on a scale of 10^{-10}. The corresponding frequency is 6GHz. We thus suppose that the space can be thought as being isotropic in the approximate computation to a certain target.

It is seen from Fig. 5 the wavenumber is affected by the shape of target also to the same extent of the observing azimuth. The results in other azimuths are nearly the same as those shown in Fig.5.

It is worth pointing out that a, b, c in the former process are merely dimensionless factors (they do not have length unit). They must be positive and real numbers. The obtained results are the same as those in a general coordinate system when all the scale-factors are equal to one, which implies the validity

of the obtained results.

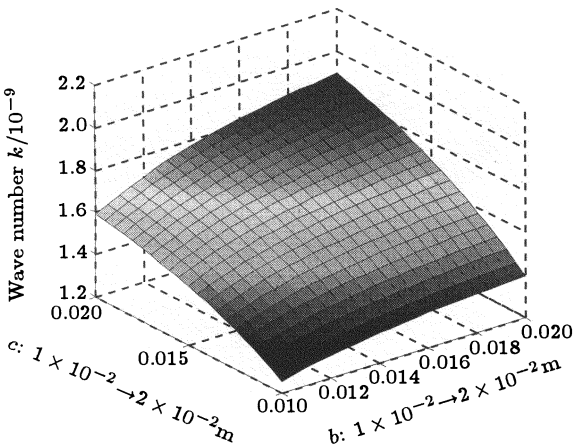


Fig.5. Size of wavenumber versus shape, f=6GHz, $\theta=\pi/4$, $\phi=\pi/3$.

The result in Ref.[7] and the simulation result of scale transformation are compared in Fig.6, which indicates the change of the squared scattering field with the distance. This difference in Fig.6 is about 10^{-18}, which should produce no effects on practical applications. The Rayleigh scattering on the other hand is a kind of good approximation at very low frequencies.

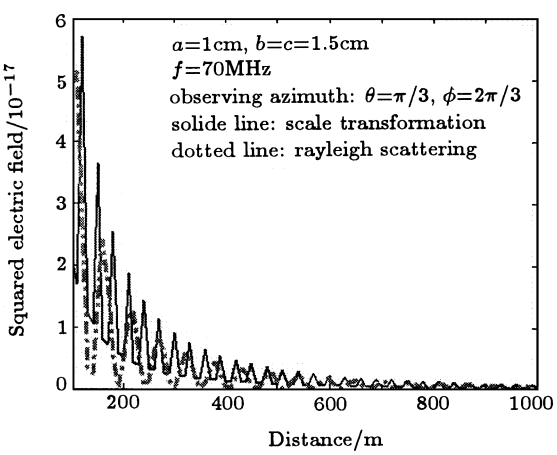


Fig.6. Squared scattering field versus distance.

Assume there exists a conductive ellipsoid with semi-axes a, b, b in the coordinate system Σ, as shown in Fig.1. An electromagnetic plane wave polarizing in the x direction and propagating in the z direction irradiates an ellipsoid. In the system Σ', we obtain

$$E'_{0x}\mathrm{e}^{-\mathrm{j}k'z'}=aE_{0x}\mathrm{e}^{-\mathrm{j}k'z'},$$
$$k'=ck.$$

In the system Σ', the ellipsoid changes into a sphere with a radius of 1meter, lying at the original position. When the plane wave irradiates the sphere, the scattering fields[9] are as follows:

$$\begin{aligned}
E_{\theta's}=&\frac{\mathrm{j}aE_{0x}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}\cos\varphi'\\
&\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[c'_n\tau_n+d'_n\pi_n,\right]\\
=&\frac{\mathrm{j}aE_{0x}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}\cos\varphi' s_2(\theta'),\\
E_{\varphi's}=&\frac{-\mathrm{j}aE_{0x}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}\sin\varphi'\\
&\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\left[c'_n\pi_n+d'_n\tau_n\right],\\
=&\frac{-\mathrm{j}aE_{0x}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}\sin\varphi' s_1(\theta'),\\
\tau_n=&\frac{\mathrm{d}P_n^1(\cos\theta')}{\mathrm{d}\theta'},\\
\pi_n=&\frac{P_n^1(\cos\theta')}{\sin\theta'},
\end{aligned}\tag{26}$$

where

$$c'_n=-\frac{\hat{J}'_n(k'\times1)}{\hat{H}'_n(k'\times1)}=-\frac{\hat{J}'_n(kc)}{\hat{H}'_n(kc)},$$
$$d'_n=-\frac{\hat{J}_n(k'\times1)}{\hat{H}_n(k'\times1)}=-\frac{\hat{J}_n(kc)}{\hat{H}_n(kc)}.$$

Utilizing the inverse transformation of (11b), the fields in the Σ system are given by

$$\begin{aligned}
E_{\theta sT}=&\frac{g}{qc}\frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}s_2(\theta')(aE_{0x}\cos\varphi'),\\
E_{\varphi sT}=&\frac{\cos\theta\sin2\varphi}{2cqg}\frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}s_2(\theta')(aE_{0x}\cos\varphi')\\
&-\frac{1}{abg}\frac{\mathrm{j}\mathrm{e}^{-\mathrm{j}k'r'}}{k'r'}s_1(\theta')(aE_{0x}\sin\varphi').
\end{aligned}\tag{27}$$

The definitions of g and q can be found in expression (5). The result of the formula (27) and the result of mixed arithmetic are both shown in Fig.6. In this arithmetic, the test function is a δ-function and the radix function is the expression (26), so the c_n and d_n are the coefficients to be determined. In this method we use the system identification to determine the coefficients. The errors are minimized. Therefore, the method used is a combination of momentum method and system identification.

In Fig.7, the scattering field is a backward field and the incident wave polarizes in the x direction. Other parameters are shown in the figure. In simulation, the series is truncated at n=6. We conclude from the results that the two methods are identical in computing the scattering fields and the error is caused by the truncation of series.

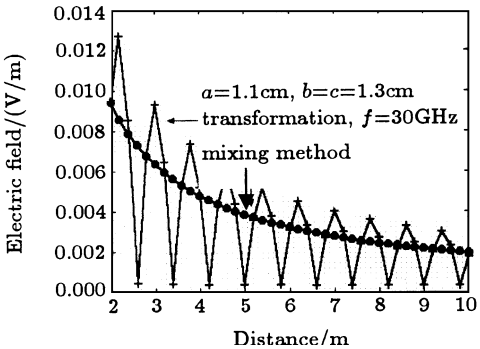


Fig.7. Scattering by a conductive ellipsoid.

5. Electromagnetic field in elliptical cylinder system

In reality, many scattering objects such as trees and flyers that can be described as elliptical cylinders lie across the propagating direction of electromagnetic wave. Thus investigating the characteristics of electromagnetic scattering by an elliptical cylinder is of great value in engineering.[9,10] If a scattering problem about an elliptical cylinder is converted into that about a cylinder, we can avoid many complex computations, and obtain an analytical solution by utilizing the references available.

Assuming the symmetrical axis of an elliptical cylinder is parallel to the z-axis, the scale-transformation is made only in the x–y plane. For an elliptical cylinder, it can be supposed that

$$\begin{aligned} &c = 1, \\ &a > 0, b > 0. \end{aligned} \tag{28}$$

All the obtained results and conclusions are valid and Maxwell equations have the same forms as those in a general situation. The transforming matrices T and M can be written as

$$T^{-1}(a,b,c) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$M^{-1} = \begin{bmatrix} b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & ab \end{bmatrix}.$$

The relations for the relevant quantities are the same. As a matter of fact, the expressions derived before are universal.

6. Conclusion

The scale-transformation for coordinate is introduced. The relations between the spherical coordinate and the differential functor are presented. The transformation relations of electric charge density and electric current density vector are developed based on the form of the conservation law of electric charge being uncharged in different coordinate systems. The scale-transformation of electromagnetic field is made with the form of Maxwell equations unchanged in different coordinate system. The boundary conditions are also investigated. A novel approach to further studying the electromagnetic scattering subjects, resonant problems relative to the objects such as an elliptical sphere and an elliptical cylinder is presented. A theoretical basis is established for utilizing the results available for a sphere and a cylinder to investigate the scattering problems. Although the space is anisotropic after the scale transforming, simulation in a millimeter wave band shows that the wavenumber does not change greatly with the scale factors and the azimuth, which implies that the space can be considered to be isotropic in engineering. The scale invariants are obtained and their physical meanings are presented.

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