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Analytical theory of DC SQUID with a resistively shunted inductance driven by thermal noises

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An analytical expression for the stationary probability distribution of the DC superconducting quantum interference device (SQUID) with a resistively shunted inductance driven by thermal noise is derived from the two-dimensional Fokker–Planck equation. The effects on the SQUID characteristics subject to a large thermal fluctuation with a noise parameter $\Gamma > 0.20$ are discussed by taking into account the thermal noise in the accuracy of numerical simulation. This theory is valid for a reduced inductance $\beta \leq 1$. The analytical formulae for the SQUID characteristics, e.g. the circulating current, the average voltage and the voltage modulation, are obtained and discussed. The theory shows that the voltage modulation increases with the shunted inductance more efficiently for a large inductance parameter β and small fluctuation parameter Γ.

Keywords: noise, Fokker–Planck equation, SQUID, shunted
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1. Introduction

The DC superconducting quantum interference device (SQUID) made up of the high Tc superconductors has been extensively studied both experimentally and theoretically. Many numerical simulations have been performed for optimizing the SQUID's characteristics. But there exists a significant discrepancy between experimental results and numerical simulations for the high T_c DC SQUID.[1−3] The possible reason is that one is lacking in powerful computer software for solving the stochastic differential equations accurately, which is a feature in numerical simulation for high T_c DC SQUIDs. Also, analytical theory could only poorly describe the SQUID's characteristics before the advent of Chesca's theory (1998).[3] This is one of the most important problems, which seriously hinders the high T_c SQUID in applications. Recently, more and more attention has been paid to analytical theory based on solving the 2D Fokker–Planck equations.[3,4] From most of these theories the approximate solutions have been obtained through expanding the distribution function with a small parameter.[3−6] Chesca expanded the distribution function with a small reduced inductance β[3] while Greenberg obtained the solution with a large inductance.[4,5] Recently, an analytical theory has been developed by Jia *et al*[7] which is not restricted by a small reduced inductance and low value of thermal fluctuation.

In the past 30 years, many sorts of SQUIDs have been designed theoretically and performed experimentally.[1,5,6,8] A DC SQUID with a resistively shunted inductance has been proposed and studied[8−15] for developing the coupling between the SQUID and input coil. But as mentioned before, all these researches were restricted in a small thermal fluctuation (TF) with a noise parameter $\Gamma < 0.25$ and performed through numerical simulation. There exists no analytical theory which is developed on one of these models, however it is important to describe the characteristics of the high T_c DC SQUID analytically.

The aim of this paper is to extend Chesca's theory to the case of DC SQUID with a resistively shunted inductance. Here we present the most important SQUID characteristics, such as circulating current, average voltage and voltage modulation. This analytical theory is developed in first order approximation with the DC SQUID reduced inductance β considered as a small parameter, so that the results are valid only

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for $\beta \leq 1$. The theory is applicable in the case of either a small TF or a large TF.

2. Theoretical model

The behaviour of the DC SQUID with a resistively shunted inductance is described by the following set of equations:[8,9]

$$\begin{aligned} & I_{\mathrm{c}} \sin\theta_1 + I_{\mathrm{c}} \sin\theta_2 \\ =& I_{\mathrm{b}} + I_{N_1} + I_{N_2} - (V_1 + V_2)/R, \end{aligned} \tag{1}$$

$$\begin{aligned} & I_{\mathrm{c}} \sin\theta_1 - I_{\mathrm{c}} \sin\theta_2 \\ =& \frac{2\Phi_x}{L} + I_{N_1} - I_{N_2} + 2I_{N_3} \\ & - \frac{(1+2\gamma)}{R}(V_1 - V_2) - \frac{\Phi_0}{\pi L}(\theta_1 - \theta_2), \end{aligned} \tag{2}$$

where θ_k is the phase difference across junction, Φ_x is the flux applied at a low frequency and to be measured, I_{b} is the constant bias current, I_{c} is the critical current of the two junctions, $\gamma = R/R_{\mathrm{d}}$ is the damping parameter and R_{d} is the damping resistance shunted in the SQUID loop L. It is assumed that the input noise in the DC SQUID can be approximated by the Johnson noise in the three shunt resistors; their corresponding independent Nyquist noise currents $I_{N_k}(t)$s respectively are

$$\langle I_{N_k}(t) I_{N_m}(t') \rangle = \frac{2k_{\mathrm{B}}T}{R}\delta_{km}\delta(t-t'),$$

$$\langle I_{N_3}(t) I_{N_3}(t') \rangle = \frac{2\gamma k_{\mathrm{B}}T}{R}\delta(t-t'),$$

$$\langle I_{N_k}(t) I_{N_3}(t') \rangle = 0,$$

where k_{B} is the Boltzmann constant, T is the absolute temperature, δ_{km} is the Kronecker delta, k and m are 1 and 2 respectively, and $\delta(t)$ is the Dirac delta function of time t.

From Eqs.(1) and (2), we introducing the new variables $\phi = (\theta_1 - \theta_2)/2$ and $\varphi = (\theta_1 + \theta_2)/2$, we obtain

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}t} =& -\frac{2R}{(1+2\gamma)L}(\phi - \phi_x) \\ & - \frac{\omega_{\mathrm{c}}}{1+2\gamma}\sin\phi\cos\varphi + v_N(t), \end{aligned} \tag{3}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{i}\omega_{\mathrm{c}}}{2} - \omega_{\mathrm{c}}\sin\varphi\cos\phi + w_N(t), \tag{4}$$

where $v_N(t) = [I_{N_1}(t) - I_{N_2}(t) + 2I_{N_3}(t)]\dfrac{\pi R}{(1+2\gamma)\Phi_0}$, $w_N(t) = [I_{N_1}(t) + I_{N_2}(t)]\dfrac{\pi R}{\Phi_0}$, $\omega_{\mathrm{c}} = 2\pi R I_{\mathrm{c}}/\Phi_0$, and $i = I_{\mathrm{b}}/I_{\mathrm{c}}$, $\phi_x = \pi\Phi_x/\Phi_0$. $v_N(t)$ and $w_N(t)$ are still Gaussian white noises with zero mean. And there exist

$$\begin{aligned} \langle v_N(t) v_N(t') \rangle =& \frac{1}{1+2\gamma}\frac{4\pi^2 R k_{\mathrm{B}}T}{\Phi_0^2}\delta(t-t'), \\ \langle w_N(t) w_N(t') \rangle =& \frac{4\pi^2 R k_{\mathrm{B}}T}{\Phi_0^2}\delta(t-t'), \\ \langle v_N(t) w_N(t') \rangle =& 0. \end{aligned} \tag{5}$$

The Fokker–Planck equation that corresponds to Eqs.(3) and (4) is

$$\begin{aligned} & \frac{L}{2R}\frac{\partial}{\partial t}P + \frac{\pi\beta i}{4}\frac{\partial}{\partial\varphi}P - \frac{1}{1+2\gamma}\frac{\partial}{\partial\phi}[(\phi - \phi_x)P] \\ & - \alpha\left[\frac{\partial^2}{\partial\varphi^2} + \frac{1}{1+2\gamma}\frac{\partial^2}{\partial\phi^2}\right]P \\ =& \frac{\pi\beta}{2}\left[\frac{\cos\varphi}{1+2\gamma}\frac{\partial}{\partial\phi}(\sin\phi P) + \cos\phi\frac{\partial}{\partial\varphi}(\sin\varphi P)\right], \end{aligned} \tag{6}$$

where $\alpha = \pi^2 k_{\mathrm{B}}TL/\Phi_0^2$, $\beta = 2LI_{\mathrm{c}}/\Phi_0$, and $P(t,\phi,\varphi)$ is the probability density of finding the DC SQUID in a state (ϕ,φ) at time t. As can be inferred from Eqs.(3) and (4), $P(t,\phi,\varphi)$ is 2π periodic in φ. Then, because the total time-averaged voltage $\overline{V} = \overline{v}I_{\mathrm{c}}R$ across the DC SQUID oscillates with the period $T_0 = \Phi_0/(I_{\mathrm{c}}R)\overline{v}$, $P(t,\phi,\varphi)$ should also have the same periodicity with t.[4]

Here we try to solve Eq.(6) by considering β as a small parameter. As a result we can find the probability distribution in the form

$$P = P_0 + \frac{\beta}{2}P_1, \tag{7}$$

where P_0 and P_1 are respectively in the zero and first order approximations over β in Eq.(6).

We obtain the following hierarchy of Eq.(6) respectively for P_0 and P_1:

$$\begin{aligned} & \frac{L}{2R}\frac{\partial}{\partial t}P_0 + \frac{\pi\beta i}{4}\frac{\partial}{\partial\varphi}P_0 - \frac{1}{1+2\gamma}\frac{\partial}{\partial\phi}[(\phi - \phi_x)P_0] \\ & - \alpha\left[\frac{\partial^2}{\partial\varphi^2} + \frac{1}{1+2\gamma}\frac{\partial^2}{\partial\phi^2}\right]P_0 = 0, \end{aligned} \tag{8}$$

$$\begin{aligned} & \frac{L}{2R}\frac{\partial}{\partial t}P_1 + \frac{\pi\beta i}{4}\frac{\partial}{\partial\varphi}P_1 - \frac{1}{1+2\gamma}\frac{\partial}{\partial\phi}[(\phi - \phi_x)P_1] \\ & - \alpha\left[\frac{\partial^2}{\partial\varphi^2} + \frac{1}{1+2\gamma}\frac{\partial^2}{\partial\phi^2}\right]P_1 \\ =& \pi\frac{\cos\varphi}{1+2\gamma}\frac{\partial}{\partial\phi}(\sin\phi P_0) + \pi\cos\phi\frac{\partial}{\partial\varphi}(\sin\varphi P_0). \end{aligned} \tag{9}$$

The normalized solution to Eq.(8) is (for details see Ref.[3])

$$P_0 = \sqrt{\frac{1}{(2\pi)^3\alpha}}\exp\left[-\frac{1}{2\alpha}(\phi - \phi_x)^2\right]. \tag{10}$$

It can be seen from Eq.(10) that P_0 is independent of φ. The right-hand side of Eq.(9) can be written as

$$\pi\left[\frac{\cos\varphi}{1+2\gamma}\frac{\partial}{\partial\phi}(\sin\phi P_0)+\cos\phi\frac{\partial}{\partial\varphi}(\sin\varphi P_0)\right] =\mathrm{Re}(\mathrm{e}^{\mathrm{j}\varphi})f(\phi), \tag{11}$$

where

$$f(\phi)=\pi\left(\frac{2+2\gamma}{1+2\gamma}\cos\phi P_0+\frac{1}{1+2\gamma}\sin\phi\frac{\partial}{\partial\phi}P_0\right), \tag{12}$$

$\mathrm{j}^2=-1$, and Re denotes the real part of the complex expression.

As pointed out, the probability distribution is 2π periodic in φ, and we can expand $P_1(t,\phi,\varphi)$ into a series of $\cos(m\varphi)$ and $\sin(m\varphi)$ terms as follows, where the m are integers:

$$P_1(t,\phi,\varphi)=\mathrm{Re}\left[\sum_{m=0}^{\infty}\mathrm{e}^{\mathrm{j}m\varphi}P_{1m}^*(t,\phi)\right]. \tag{13}$$

Here $P_{1m}^*(t,\phi)$ each are a complex function, so they each have both real (Re) and imaginary (Im) components. Considering Eqs.(11) and (13), from Eq.(9) we can obtain

$$\frac{L}{2R}\frac{\partial}{\partial t}P_{11}^*+\frac{\pi\beta\mathrm{ij}}{4}P_{11}^*-\frac{1}{1+2\gamma}\frac{\partial}{\partial\phi}[(\phi-\phi_x)P_{11}^*] -\alpha\left[\frac{1}{1+2\gamma}\frac{\partial^2}{\partial\phi^2}-1\right]P_{11}^*=f(\phi), \tag{14}$$

$$\frac{L}{2R}\frac{\partial}{\partial t}P_{1m}^*+\frac{\pi\beta ijm}{4}P_{1m}^*-\frac{1}{1+2\gamma}\frac{\partial}{\partial\phi}[(\phi-\phi_x)P_{1m}^*] -\alpha\left[\frac{1}{1+2\gamma}\frac{\partial^2}{\partial\phi^2}-m^2\right]P_{1m}^*=0\quad m\neq 1. \tag{15}$$

3. Statistical characteristics of DC SQUID

3.1. Circulating current

The purpose of this section is to obtain the statistical average (denoted by $<\ J\ >$) of the time-averaged (denoted by $\bar{J}$) circulating current J in the DC SQUID. From Eq.(3) we can obtain

$$\frac{<\bar{J}>}{I_\mathrm{c}}=-<\overline{\sin\phi\cos\varphi}> =-\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\phi\cos\varphi P(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi, \tag{16}$$

where the phase difference ϕ has a statistically independent constant average value ($<\overline{\mathrm{d}\phi/\mathrm{d}t}>=0$) with an initial transient assumed. The assumption $\overline{\mathrm{d}\phi/\mathrm{d}t}=0$ is valid for the TF being not too large. Therefore, the results of this section are applicable in the case of small TF.

The purpose of the present calculation is not to solve Eqs.(14) and (15) explicitly but to find the average from Eq.(16). Therefore, the integral

$$I_1(t)=\int_{-\infty}^{\infty}\sin\phi P_{1m}^*(t,\phi)\mathrm{d}\phi \tag{17}$$

is of importance in the following. As a consequence, we make a change as follows:

$$\Lambda_{1m}^*(t,x)=\int_{-\infty}^{\infty}\sin(x\phi'+\phi_x)P_{1m}^*(t,\phi')\mathrm{d}\phi', \tag{18}$$

where $\phi'=\phi-\phi_x$. Then, multiplying Eqs.(14) and (15) by $\sin(x\phi'+\phi_x)$ and integrating over ϕ' from $-\infty$ to $+\infty$ we obtain the differential equations satisfied by Λ_{1m}^*:

$$\frac{L}{2R}\frac{\partial}{\partial t}\Lambda_{11}^*+\frac{1}{1+2\gamma}x\frac{\partial}{\partial x}\Lambda_{11}^* +\Lambda_{11}^*\left[\frac{\alpha}{1+2\gamma}x^2+\alpha+\frac{\pi\beta\mathrm{ij}}{4}\right]=F(x), \tag{19}$$

$$\frac{L}{2R}\frac{\partial}{\partial t}\Lambda_{1m}^*+\frac{1}{1+2\gamma}x\frac{\partial}{\partial x}\Lambda_{1m}^* +\Lambda_{1m}^*\left[\frac{\alpha}{1+2\gamma}x^2+\alpha m+\frac{\pi\beta ijm}{4}\right]=0,\quad m\neq 1, \tag{20}$$

where

$$F(x)=\int_{-\infty}^{\infty}\sin(x\phi'+\phi_x)f(\phi'+\phi_x)\mathrm{d}\phi' =\frac{1+2\gamma-x}{4(1+2\gamma)}\exp\left[-\frac{\alpha}{2}(1+x)^2\right]\sin 2\phi_x \tag{21}$$

with $f(\phi)$ defined in Eq.(12). Here boundary conditions of the Fokker–Planck equation (FPE)

$$P_1^*|_{\phi=\pm\infty}=0,\quad \frac{\partial P_1^*}{\partial\phi}|_{\phi=\pm\infty}=0$$

and

$$P_0|_{\phi=\pm\infty}=0 \tag{22}$$

have been used.

The probability distribution is periodic in t (the period is $T_0=2\pi/\omega$), so that we can expand $\Lambda_{1m}^*(t,x)$ into a Fourier series

$$\Lambda_{1m}^*(t,x)=\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}n\omega t}\Lambda_{1mn}^+(x). \tag{23}$$

Combining Eqs.(19), (20) and (23) yields the fol-

lowing differential equations satisfied by $\Lambda^{+}_{1mn}(x)$:

$$\frac{1}{1+2\gamma}x\frac{\mathrm{d}}{\mathrm{d}x}\Lambda^{+}_{11n} + \Lambda^{+}_{11n}\left[\frac{\alpha}{1+2\gamma}x^2+\Upsilon_{1n}\right]=F(x), \tag{24}$$

$$\frac{1}{1+2\gamma}x\frac{\mathrm{d}}{\mathrm{d}x}\Lambda^{+}_{1mn} + \Lambda^{+}_{1mn}\left[\frac{\alpha}{1+2\gamma}x^2+\Upsilon_{mn}\right]=0,\quad m\neq 1, \tag{25}$$

where $\Upsilon_{mn}=\alpha m^2+(\pi\beta i m/4+\omega Ln/2R)\mathrm{j}$.

Combining Eqs.(21), (24) and (25) gives the values taken by $\Lambda^{+}_{1mn}(x)$ for $x=1$:

$$\Lambda^{+}_{11n}(x=1)=[(1+2\gamma)I(\Upsilon'_{1n}-1,-\alpha) - I(\Upsilon'_{1n},-\alpha)]\frac{\mathrm{e}^{-\alpha}}{4}\sin 2\phi_x, \tag{26}$$

$$\Lambda^{+}_{1mn}(x=1)=C_{mn}\mathrm{e}^{-\alpha/2},\quad m\neq 1, \tag{27}$$

where $I(A,\alpha)=\int_0^1 y^A\mathrm{e}^{\alpha y}\mathrm{d}y$ and $\Upsilon'_{1n}=(1+2\gamma)\Upsilon_{1n}$.

It is easy to derive a useful recurrence formula for these integrals $I(A-1,\alpha)=[\mathrm{e}^{\alpha}-\alpha I(A,\alpha)]/A$, so Eq.(26) can be written as

$$\Lambda^{+}_{11n}(x=1)=\frac{\mathrm{e}^{-\alpha}}{4}\sin 2\phi_x\left[\frac{1}{\Upsilon_{1n}}\mathrm{e}^{-\alpha}-\left(1-\frac{\alpha}{\Upsilon_{1n}}\right)I(\Upsilon'_{1n},-\alpha)\right]. \tag{28}$$

Combining Eqs.(10), (13), (16) and (23) yields

$$\begin{aligned}
&<\overline{\sin\phi\cos\varphi}>\\
&=\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\phi\cos\varphi P(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi\\
&=\frac{\beta}{2}C_1\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\phi\cos\varphi P_1(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi\\
&=\frac{\beta}{2}C_1\int_0^{2\pi}\cos\varphi\\
&\quad\times\mathrm{Re}\left[\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}m\varphi}\mathrm{e}^{\mathrm{j}n\omega t}\Lambda^{+}_{1mn}(x=1)\right]\mathrm{d}\varphi\\
&=\frac{\beta}{2}C_1\int_0^{2\pi}\overline{\cos\varphi\mathrm{Re}\left[\mathrm{e}^{\mathrm{j}m\varphi}\sum_{m=1}^{\infty}\Lambda^{+}_{1m0}(x=1)\right]}\mathrm{d}\varphi\\
&=\frac{\beta}{2}C_1\frac{1}{2}\mathrm{Re}\left[\Lambda^{+}_{110}(x=1)\right].
\end{aligned} \tag{29}$$

The constant in Eq.(29) appears from the normalization condition, and therefore is defined by $1/C_1=\int_{-\infty}^{\infty}\int_0^{2\pi}P(t,\phi,\varphi)\mathrm{d}\varphi\mathrm{d}\phi$.

Finally, combining Eqs.(16), (26) and (29), the statistical average of the time-averaged circulating current in the DC SQUID is given by

$$\frac{<\bar{J}>}{I_{\mathrm{c}}}=\frac{\pi\beta}{8}S_1\sin 2\phi_x, \tag{30}$$

where

$$S_1=\frac{\mathrm{e}^{-\alpha}}{\alpha^2+\left(\frac{\pi\beta i}{4}\right)^2}\left\{\alpha\mathrm{e}^{-\alpha}-\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!}\times\frac{[2(1+2\gamma)\alpha+k+1]\left(\frac{\pi\beta i}{4}\right)^2}{[(1+2\gamma)\alpha+k+1]^2+\left[(1+2\gamma)\frac{\pi\beta i}{4}\right]^2}\right\}. \tag{31}$$

In order to obtain Eq.(30), the fact that $C_1=2\pi$ has been used. The calculation details of the normalization constant C_1 can be found in Ref.[3]. As mentioned before, the results in this section are valid only for small TF.

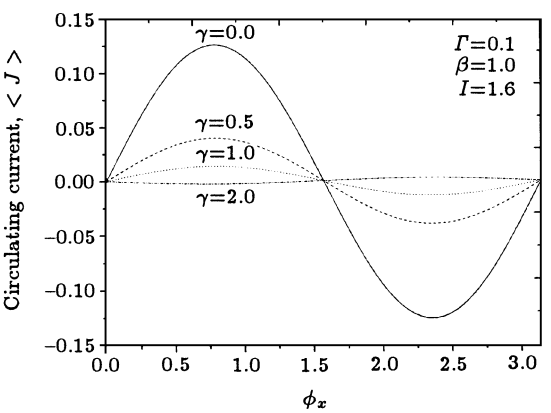


Fig.1. Circulating current $< J >$ vs applied signal flux ϕ_x as a function of γ.

Equation (30) has an identical structure with that of Chesca (Eq.(21) in Ref.[3]) in which the analyses are given in the case of $\gamma = 0$. The effect of damping resistance on the circulating current $< J >$ is shown in Fig.1, which indicates that the damping resistance suppresses the circulating current $< J >$ when $< J >> 0$ but strengthens it when $< J >< 0$. In the case of $< J >< 0$, the suppression increases with the increasing of damping resistance, and the direction of the circulating current reverses when γ reaches a critical value (see Fig.1). A reasonable explanation may be that the effect of damping resistance reduces the current difference in the circuit, which will lead to the reduction of phase difference between two junctions. From the definition of $< J >$ $\left(< J >= \frac{2}{\pi\beta} < \overline{\phi-\phi_x} > \right)$, it can be found that

the reduction of phase difference (denoted by 2ϕ) may cause the direction of the circulating current to reverse.

3.2. The SQUID Voltage and the voltage modulation

The purpose of this section is to obtain the statistical average (denoted by $< V >$) of the time-averaged (denoted by $\bar{V}$) voltage V in the DC SQUID. From Eq.(4) we can obtain

$$\begin{aligned}\frac{2\pi\langle\bar{V}\rangle}{\Phi_0} =&\left\langle\overline{\frac{\mathrm{d}}{\mathrm{d}t}\varphi}\right\rangle\\ =&\frac{i}{2}\omega_{\mathrm{c}}-\omega_{\mathrm{c}}\langle\overline{\sin\varphi\cos\phi}\rangle\\ =&\frac{i}{2}\omega_{\mathrm{c}}-\omega_{\mathrm{c}}\\ &\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\varphi\cos\phi P(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi. \quad (32)\end{aligned}$$

As shown above, in order to obtain the voltage across the SQUID first of all we need to calculate the integral in Eq.(32). The difference appears at this stage of the analysis, since now, as can be inferred from Eq.(32), the integral

$$I_2(t)=\int_{-\infty}^{\infty}\cos\phi P_{1m}^{*}(t,\phi)\mathrm{d}\phi, \quad (33)$$

rather than $I_1(t)$ is of importance, as indicated in the following. We make a change as follows:

$$\Lambda_{1m}^{*}(t,x)=\int_{-\infty}^{\infty}\cos(x\phi'+\phi_x)P_{1m}^{*}(t,\phi')\mathrm{d}\phi', \quad (34)$$

where $\phi'=\phi-\phi_x$. From Eq.(34) it is clear that for $x=1$ we can obtain the integral that we are interested in: $\Lambda_{1m}^{*}(t,x=1)=I_2(t)$. Then, multiplying Eqs.(14) and (15) by $\cos(x\phi'+\phi_x)$ and integrating with respect to ϕ' from $-\infty$ to $+\infty$, we find that Λ_{1m}^{*} satisfies Eqs.(19) and (20) with the only change that $F(x)$ now is in the following form:

$$\begin{aligned}F(x)=&\int_{-\infty}^{\infty}\cos(x\phi'+\phi_x)f(\phi'+\phi_x)\mathrm{d}\phi'\\ =&\frac{1}{4(1+2\gamma)}\Big\{(1+2\gamma-x)\\ &\times\exp\left[-\frac{\alpha}{2}(1+x)^2\right]\cos 2\phi_x\\ &+(1+2\gamma+x)\exp\left[-\frac{\alpha}{2}(1-x)^2\right]\Big\} \quad (35)\end{aligned}$$

with $f(\phi)$ defined in Eq.(12). Further, as shown before, introducing the functions $\Lambda_{1mn}^{+}(x)$ defined by Eq.(23) and using $F(x)$ defined by Eq.(35), we can obtain the values taken by $\Lambda_{1mn}^{+}(x)$ at $x=1$:

$$\begin{aligned}\Lambda_{11n}^{+}(x=1)=&\{[(1+2\gamma)I(\Upsilon_{1n}'-1,-\alpha)\\ &-I(\Upsilon_{1n}',-\alpha)]\cos 2\phi_x\\ &-[(2\gamma+1)I(\Upsilon_{1n}'-1,\alpha)\\ &+I(\Upsilon_{1n}',\alpha)]\}\frac{\mathrm{e}^{-\alpha}}{4}, \quad (36)\end{aligned}$$

$$\Lambda_{1mn}^{+}(x=1)=C_{mn}\mathrm{e}^{-\alpha/2}, \quad m\neq 1, \quad (37)$$

where $I(A,\alpha)=\int_0^1 y^A\mathrm{e}^{\alpha y}\mathrm{d}y$, $\Upsilon_{1n}'=(1+2\gamma)\Upsilon_{1n}$.

Combining Eqs.(10), (13), (32) and (36) yields

$$\begin{aligned}\langle\overline{\sin\varphi\cos\phi}\rangle=&\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\varphi\cos\phi P(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi\\ =&\frac{\beta}{2}C_1\int_{-\infty}^{\infty}\int_0^{2\pi}\overline{\sin\varphi\cos\phi P_1(t,\phi,\varphi)}\mathrm{d}\varphi\mathrm{d}\phi\\ =&\frac{\beta}{2}C_1\int_0^{2\pi}\overline{\sin\varphi\mathrm{Re}\left[\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{j}m\varphi}\mathrm{e}^{\mathrm{j}n\omega t}\Lambda_{1mn}^{+}(x=1)\right]}\mathrm{d}\varphi\\ =&\frac{\beta}{2}C_1\int_0^{2\pi}\sin\varphi\mathrm{Re}\left[\mathrm{e}^{\mathrm{j}m\varphi}\sum_{m=1}^{\infty}\Lambda_{1m0}^{+}(x=1)\right]\mathrm{d}\varphi\\ =&\frac{\beta}{2}C_1\frac{1}{2}\mathrm{Re}\left[\mathrm{j}\Lambda_{110}^{+}(x=1)\right]. \quad (38)\end{aligned}$$

As mentioned before, the result depends on integral $I(A,\alpha)$, where A is a complex number.

Finally, combining Eqs.(32), (36) and (38), the statistical average of the time-averaged voltage ($V=I_{\mathrm{c}}Rv$) across the DC SQUID is given by

$$v=\frac{1}{2}i\left\{1-\frac{\mathrm{e}^{-\alpha}\left(\frac{\pi\beta}{4}\right)^2}{\alpha^2+\left(\frac{\pi\beta i}{4}\right)^2}\left[\left(\mathrm{e}^{-\alpha}+S_-\right)\cos 2\phi_x+\left(\mathrm{e}^{\alpha}-S_+\right)\right]\right\}, \quad (39)$$

where

$$S_{\pm} = \sum_{k=0}^{\infty} \frac{(\pm\alpha)^k}{k!} \frac{\alpha[(1+2\gamma)\alpha + k + 1] - (1+2\gamma)\left(\frac{\pi\beta i}{4}\right)^2}{[(1+2\gamma)\alpha + k + 1]^2 + \left[(1+2\gamma)\frac{\pi\beta i}{4}\right]^2}. \tag{40}$$

The normalization constant C_1 is also chosen to be equal to 2π. This expression is valid for $v \geq 0$ and for the case of either a small TF or a large TF. However, Eq.(39) is not applicable in the noise free case $\Gamma = 0$.

The voltage modulation is easily obtained from Eq.(39) as indicated below:

If $v_{\min}(i) \geq 0$,

$$\Delta v = \frac{i\mathrm{e}^{-\alpha}\left(\frac{\pi\beta}{4}\right)^2}{\alpha^2 + \left(\frac{\pi\beta i}{4}\right)^2}\left(\mathrm{e}^{-\alpha} + S_-\right);$$

If $v_{\min}(i) < 0$,

$$\Delta v = \frac{1}{2} i \left\{ 1 + \frac{\mathrm{e}^{-\alpha}\left(\frac{\pi\beta}{4}\right)^2}{\alpha^2 + \left(\frac{\pi\beta i}{4}\right)^2}[\left(\mathrm{e}^{-\alpha} + S_-\right) - (\mathrm{e}^{\alpha} - S_+)] \right\}. \tag{41}$$

The curves for voltage versus external flux are shown in Fig.2 for different values of damping parameter γ and the noise parameter Γ. In Fig.2(a), it is presented that the voltage modulation increases with damping parameter γ, and efficiency of the increase decreases with the increasing of γ in the case of a relatively large reduced inductance ($\beta = 1$) and a small noise ($\Gamma = 0.2$), which seems to be the optimum condition in the presence of damping resistance. For the high noise ($\Gamma = 1.5$) and the same β (shown in Fig.2(b)), the voltage–flux curves descend but the voltage modulation does not change. In the case of a small reduced inductance ($\beta = 0.1$), the voltage modulation descends slightly (as shown in Fig.2(c)).

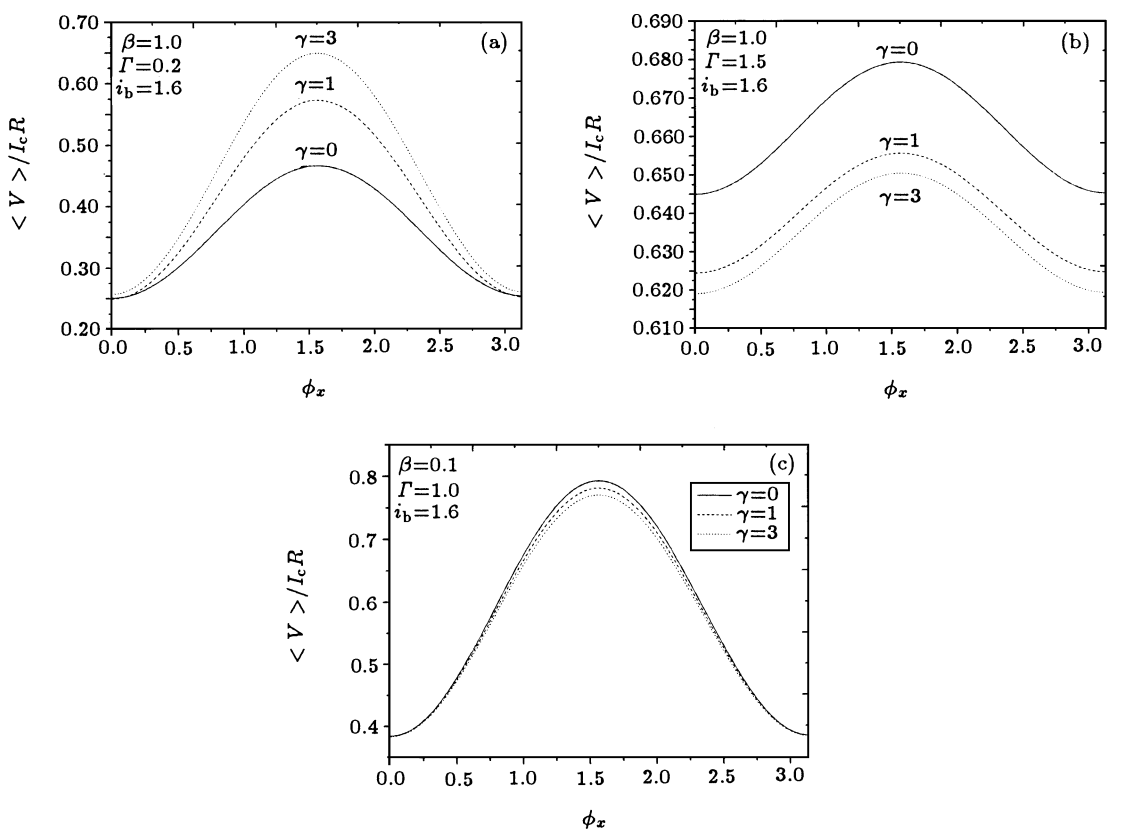


Fig.2. Average voltage $< V > /I_cR$ vs applied signal flux ϕ_x as a function of γ. (a) $\beta = 1.0$, $\Gamma = 0.2$ (b) $\beta = 1.0$, $\Gamma = 1.5$ (c) $\beta = 0.1$, $\Gamma = 1.0$.

The dependence of normalized voltage modulation on the damping resistance for several values of inductance parameter β is shown in Fig.3. It is clearly seen that for a relatively small inductance ($\beta = 0.1$) the damping parameter reduces the voltage modulation whereas for a relatively large inductance ($\beta > 0.5$) the damping parameter increases the voltage modulation, especially in the case of small noise. This results show that the presence of damping resistance is more useful and efficient for DC SQUID with a large inductance in the case of a small TF, which can improve the coupling efficiency. The results are shown to be in good agreement with those in Ref.[9].

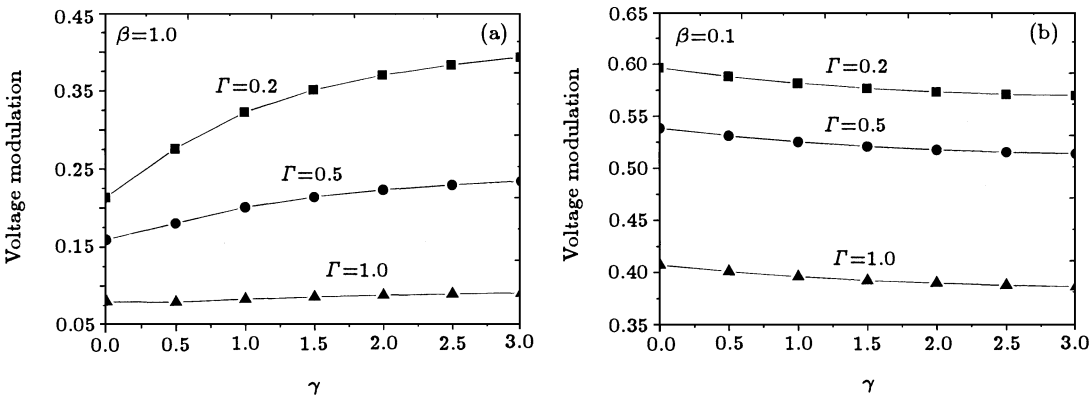


Fig.3. Voltage modulation vs damping parameter γ as a function of noise parameter Γ. (a) $\beta = 1.0$, (b)$\beta = 0.1$.

4. Conclusions

In this paper, we have presented an analytical theory of the DC SQUID with a resistively shunted inductance in the presence of thermal fluctuation. The analytical formulae for some important SQUID characteristics (e.g. the circulating current, the current–voltage characteristics and the voltage modulation) have been derived by using the distribution function. It is found that the effect of damping resistance can suppress the circulating current $< J >$ when $< J >> 0$ but strengthens it when $< J >< 0$. If $< J >> 0$, the direction of circulating current will reverse when γ reaches a critical value.

Comparing with the previous theoretical work, we find that the validity of the theory is determined not only by the small parameter β, but also by the value of first order. The obtained theoretical results are shown to be in good agreement with the numerical simulation results in the case of $\beta = 1$.

We have shown that the influence of shunted resistance on the voltage modulation on the level of resistance: for a relatively small inductance the damping resistance reduces the voltage modulation whereas for a relatively large inductance ($\beta > 0.5$) the damping resistance increases the voltage modulation. The results obtained in the paper seem to be important for the design of the intentional addition of a shunted inductance to a DC SQUID with a large inductance in order to achieve a large voltage modulation. The presence of a shunted inductance is only applicable in the case of a relatively large inductance parameter β and a small noise parameter Γ.

References

[1] Tesche C D and Clarke J 1977 *J. Low. Temp. Phys.* **29** 301
[2] Greenberg Y S 2002 *Physica* C **371** 156
[3] Greenberg Y S 2003 *Physica* C **383** 354
[4] Chesca B 1998 *J. Low. Temp. Phys.* **112** 165
[5] Sterck A, Weiss S and Koelle D 2002 *Appl. Phys.* A **75** 253
[6] Acebron J A, Rappel W J and Bulsara A R 2003 *Phys. Rev.* E **67** 016210
[7] Jia Y, Sun W R and Li J R 2003 *Supercond. Sci. Technol.* **16** 437
[8] Enpuku K, Muta T, Yoshida K and Irie F 1985 *J. Appl. Phys.* **58** 1916
[9] Du S W, Dai Y D and Wang S G 1999 *Acta Phys. Sin.* **48** 2364 (in Chinese)
[10] Enpuku K, Yoshida K and Kisu T 1993 *J. Appl. Phys.* **73** 7929
[11] Testa G, Sarnelli E, Pagano S, Calidonna C R and Furnari M Mango 2001 *J. Appl. Phys.* **89** 5145
[12] Testa G, Pagano S, Sarnelli E, Calidonna C R and Furnari M Mango 2001 *Appl. Phys. Lett.* **79** 2943
[13] Testa G, Granata C, Calidonna C, Russo C Di, Furnari M Mango, Pagano S, Russo M and Sarnelli E 2002 *Physica* C **368** 232
[14] Dong Zh Ch 2002 *Acta Phys. Sin.* **51** 894 (in Chinese)
[15] Lü H F, Jia Y and Huang X T 2004 *Commun. Theor. Phys.* **41** 707