Vol 14 No 3, March 2005
1009-1963/2005/14(03)/0576-07

Chinese Physics

© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

Phase transitions in a new car-following traffic flow model*

Li Li(李 莉)† and Shi Peng-Fei(施鹏飞)

Institute of Image Processing and Pattern Recognition, Shanghai Jiaotong University, Shanghai 200030, *China*

(Received 20 October 2004)

In this paper, we investigate the performance of the well-known optimal velocity car-following model (the OVM) with numerical simulation in describing the acceleration process that is induced by the motion of a leading car with a pre-specified speed profile. Results show that this model is to some extent deficient in performing this process. Modification of the OVM to overcome the deficiency is demonstrated. The linear stability for the modified model is analysed. If the linear stability condition can not be satisfied, phase transitions occur on varying the initial homogeneous headway of the traffic flow.

Keywords: car-following models, linear stability, phase transitions
PACC: 6470

1. Introduction

The similarities of the flow of traffic and the flow of a classic fluid have aroused the interest of many physicists in recent years.[1,2] Kerner and Rehborn[3] observed the phase transitions among free traffic flow, coexisting traffic flow and traffic jams. Various traffic flow models, such as car-following models and fluid dynamic models have been studied analytically using the linear stability theory and the nonlinear analysis method in order to explain these phenomena.[4,5]

In this paper, we first present a car-following model that modifies the optimal velocity car-following model (the OVM).[6] We show that this new model performs more realistically in describing the acceleration process and the deceleration process of the traffic flow. Then we study the new model with the method of linear stability theory. Analysis shows that if the linear stability condition cannot be satisfied, phase transitions occur on varying the initial homogeneous headway.

In Section 2 we analyse the performance of the OVM in describing the acceleration process induced by the motion of a leading car with a pre-specified speed profile and present the modified car-following model. In Section 3 we apply the linear stability analysis to this model. We carry out numerical simulation for the new model to study the phase transitions in Section 4. Section 5 summarizes.

2. Analysis on car-following models

Assuming a series of cars is moving along a single lane without passing, bottleneck and inflow. The cars are given uniform car characteristics and the length of car is ignored. The OVM is described by the following differential equation:

$$\ddot{x}_n(t) = \alpha(V(\Delta x_n(t)) - \dot{x}_n(t)). \tag{1}$$

Here $x_n(t)$ denotes the position of the nth car at time t, $n = 1, \ldots, N$, with N being the total number of the cars. $\Delta x_n(t) = x_{n-1}(t) - x_n(t)$ is the headway of the nth car, $\ddot{x}_n(t)$ denotes the acceleration of the car n. α is the reciprocal of the relaxation time $T_{\rm r}$. $V(\Delta x_n(t))$ represents the optimal velocity function of the nth car decided by its headway. The optimal velocity function possesses the following properties: it is a monotonically increasing function and it has an upper bound $v_{\max}$. In this paper the optimal velocity function is given by[7]

$$V(\Delta x_n) = \frac{v_{\max}}{2}(\tanh(\Delta x_n - h_{\rm c}) + \tanh(h_{\rm c})) \tag{2}$$

*Project supported by the State Key Development Programme for Basic Research of China (Grant No G1998030408).
††E-mail: joujou@sjtu.edu.cn

with the maximal speed $v_{\max}$=2m·s^{-1} and the safety distance h_c=5m.

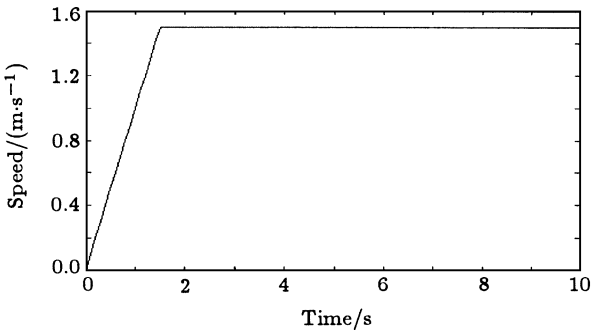


Fig.1. Speed profile of the leading car from zero to 1.5m·s^{-1} at a rate of 1m·s^{-2}.

The traffic disturbance induced by a leading car is created in order to analyse the performance of the OVM, supposing all the cars wait initially: for example, in the case of a series of vehicles under a traffic light. The homogeneous headway is 2m. Then we speed up the leading car 1 to 1.5m·s^{-1} at a rate of 1m·s^{-2}, followed by the succeeding vehicles moving according to Eq.(1).

The corresponding speed profile of the leading car is shown in Fig.1.

We integrate Eq.(1) with a simple Euler scheme using a time discretization of $\Delta t = 0.1s$, and translate the vehicles in each step according to

$$x_n(t+\Delta t) = x_n(t) + \dot{x}_n(t)\Delta t + \tfrac{1}{2}\ddot{x}_n(t)(\Delta t)^2. \quad (3)$$

Let N=20 and the total simulation time Γ=300s. Assuming that initially $x_1(0) = 0$ and $x_n(0) = -(n-1)h$, n=2,...,N, where h=2 is the homogeneous headway. We choose α=1s^{-1} in the simulation. Profiles of speed and acceleration of the cars moving according to Eqs.(1) and (2) are shown in Fig.2.

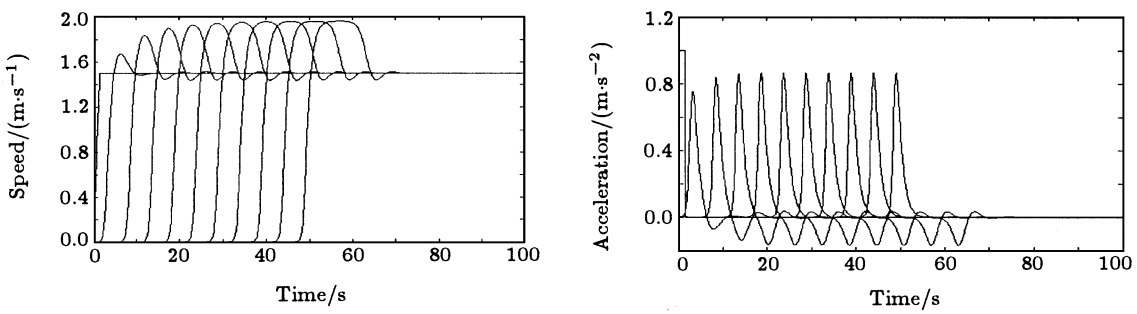


Fig.2. Speed and acceleration profiles of the OVM induced by the leading car described in Fig.1 (n=1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20). α=1.

Speed and acceleration profiles show that the OVM presents stable car-following behaviour to the disturbance (Fig.2). However, we can see that the two profiles of the OVM oscillate around the eventual equilibrium state before they become stable. As Treiber *et al*[8] pointed out, the OVM only considers the influence of deviation of the vehicle's speed from the optimal speed responding to its headway on the acceleration, which requires unrealistic small relaxation time. Therefore, the traffic flow will present undesirable behaviour if α is not large enough (the corresponding relaxation time not small enough). This can be seen clearly in Fig.3 and Table 1.

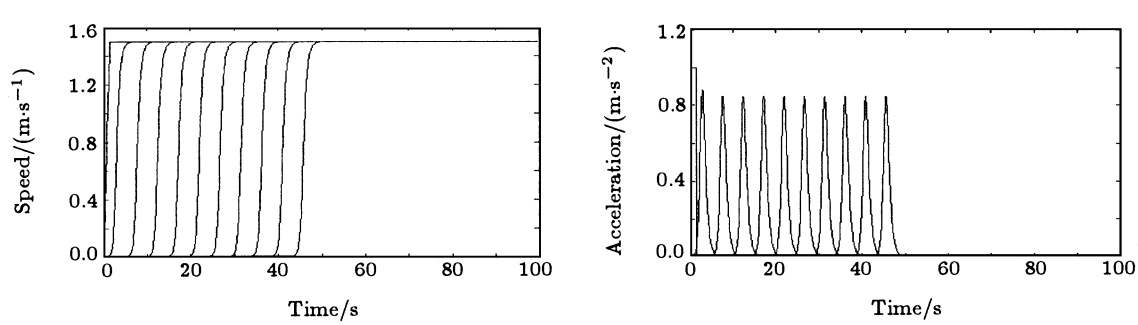


Fig.3. Speed and acceleration profiles of the OVM induced by the leading car described in Fig.1 (n=1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20). α=2.89.

Table 1. Deviations of maximal speed from the eventual equilibrium speed and the total time needed to stabilize with α=0.5, 1, 2 and 2.89 for the traffic flow moving according to the OVM induced by the leading car described in Fig.1.

α/s^{-1} [T_r]	Maximal speed deviation/m·s^{-1}	Time needed to stabilize/s
0.5 [2s]	0.50	194
1 [1s]	0.46	79.4
2 [0.5s]	0.06	55.5
2.89 [0.35s]	0	53.3

In Fig.3, we take α=2.89s^{-1} (corresponding a very small relaxation time T_r=0.35s) and the OVM can perform very well. For a clearer understanding, we list some simulation results with various α for the OVM in Table 1. The first column of the table gives the values of α. We also show the corresponding relaxation time T_r in square brackets. The second column shows the maximal speed deviations from the eventual equilibrium speed and the last column is the total time needed for the traffic flow to become stable. We can see from the list that the OVM can weaken the deviation and shorten the total time needed to stabilize with increasing α (in relation to decreasing T_r).

According to the above analysis, we modify the OVM by adding an adjusting function G:

$$\ddot{x}_n(t) = \alpha(V(\Delta x_n(t)) - \dot{x}_n(t)) + \beta G(\Delta\dot{x}_n(t), \Delta x_n(t)). \tag{4}$$

Here $0 \leq \beta \leq 1$ is the force of the adjusting term. $\Delta\dot{x}_n(t) = \dot{x}_{n-1}(t) - \dot{x}_n(t)$ is the speed difference between the (n-1)th and nth cars. $G(x, y)$ is a function of the speed difference and the headway satisfying:

(1) $G(x, y)$ is an odd function of x. That is, if the (n-1)th car moves faster than the nth car, G is positive; while if the (n-1)th car moves slower than the nth car, G is negative.

(2) G is a monotonic increasing function of x and y respectively. That is, G will increase with increasing speed difference or headway when the (n-1)th car moves faster than the nth car, while decrease with decreasing speed difference or headway when the (n-1)th car moves slower than the nth car.

In fact this adjusting term considers the effects of both speed difference and headway, which means that when a driver finds that a leading car drives faster than his, he wishes to drive at faster speed. Moreover, for larger headway, the G should take higher value to get larger acceleration. When the driver finds that the leading car is slower than his, he will slow down to avoid a collision. In this situation, the value of G should be less if the headway is less to achieve larger deceleration.

In this paper, we choose G as

$$G(\Delta\dot{x}_n(t), \Delta x_n(t)) = \begin{cases} \Delta\dot{x}_n(t)(1 + (\tanh(\Delta x_n(t) - 5))^3), \Delta\dot{x}_n(t) \geq 0; \\ \Delta\dot{x}_n(t)(1 - (\tanh(\Delta x_n(t) - 5))^3), \Delta\dot{x}_n(t) < 0. \end{cases} \tag{5}$$

The model (4) conforms to the OVM if β=0, which neglects the effects of the adjusting term. In the following, we will carry out the same simulation for this new model as aforementioned. We can choose any desired α by adjusting the values of β.

Usually realistic speed relaxation time for city traffic is of the order of 10s. Figure 4(a) shows the profiles of the traffic speed with α=0.1s^{-1} (T_r=10s), β=1s^{-1} for the new model (4) and a comparison of the speed of car 6 for the model (4) and the OVM are given in Fig.4(b). We can see that the new model performs more realistically than the OVM in this situation.

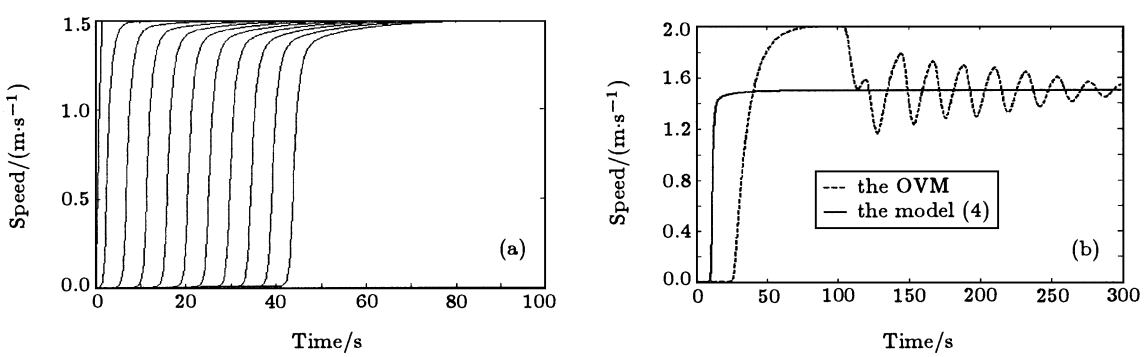


Fig.4. (a) Speed profiles of the new model with α=0.1, β=1 for the acceleration process induced by the leading car described in Fig.1 (n=1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20). (b) A comparison of the speed of car 6 for the model (4) and the OVM in this situation.

It should be noted that we could draw similar conclusions if the leading car is in the decelerating process. Figure 5 give the simulation results for the following deceleration process: supposing a series of vehicles move with an equilibrium speed $1.96\text{m}\cdot\text{s}^{-1}$ and the headway is the corresponding equilibrium one, that is h=7m. The leading car 1 slows down from $1.96\text{m}\cdot\text{s}^{-1}$ to $0.96\text{m}\cdot\text{s}^{-1}$ at a rate of $-1\text{m}\cdot\text{s}^{-2}$ and the following cars 2-N move according to Eq.(4).

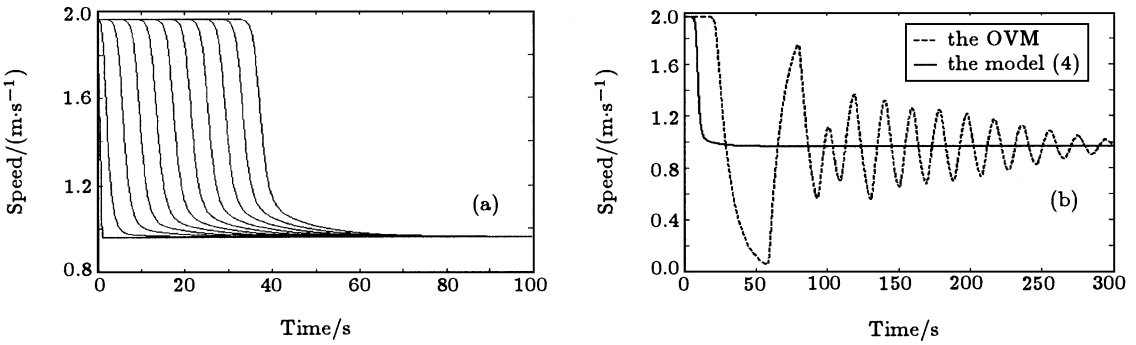


Fig.5. (a) Speed profiles of the new model with α=0.1, β=1 for the deceleration process (n=1, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20). (b) A comparison of the speed of car 6 for the model (4) and the OVM at this situation.

3. Linear stability analysis

We apply the linear stability analysis to the new model (4) with Eqs.(2) and (5). Let $x_n^0(t) = V(h)t - hn$, n=1,2... N be a uniform solution of Eq.(4), which means that all the cars move with the same headway $h = \frac{L}{N}$, where L is the length of the road. $y_n(t) = \mathrm{e}^{\mathrm{i}kn+zt}$ is a small disturbance to this equilibrium state, i.e.

$$x_n(t) = x_n^0(t) + y_n(t). \tag{6}$$

Substituting Eq.(6) into (4) and neglecting higher order terms of y_n, we have

$$z^2 + (\alpha - \beta F(\mathrm{e}^{-\mathrm{i}k} - 1))z - \alpha V'(h)(\mathrm{e}^{-\mathrm{i}k} - 1) = 0. \tag{7}$$

For the purpose of simplification, here we let $F = \begin{cases} 1 + (\tanh(h-5))^3, h \le 5; \\ 1 - (\tanh(h-5))^3, h \ge 5; \end{cases}$. The solution of the equation (7) about $\mathrm{i}k \sim 0$ is:

$$z = z_1(\mathrm{i}k) + z_2(\mathrm{i}k)^2 + ... \tag{8}$$

Here

$$z_1 = -V'(h);$$
$$z_2 = \frac{\frac{1}{2}\alpha V'(h) - (V'(h))^2 + \beta F V'(h)}{\alpha}.$$

It is known that the equilibrium solution is linear stable when $z_2 > 0$, i.e.

$$\alpha > 2(V'(h) - \beta F), \tag{9}$$

or

$$T_\mathrm{r} < \frac{1}{2(V'(h) - \beta F)}. \tag{10}$$

The condition (9) presents the phase transition curve among the free traffic, the coexisting traffic and the traffic jams. Figure 6 shows the phase diagram for the curve in the space of α against h with β=0.1, 0.2 and 0.5.

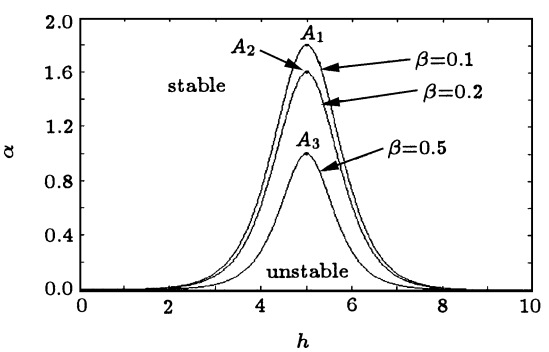


Fig.6. The phase diagram in the space of α against h with β=0.1, 0.2 and 0.5.

For a given value of β, when a small disturbance is added to the homogeneous traffic flow, this flow becomes unstable below the transition line and in time traffic jam will be formed. On the other hand, above the line, the homogeneous traffic flow is stable and remains homogeneous. If the traffic flow becomes unstable, it evolves in time into an inhomogeneous coexisting phase in which only two distinct headways depending on α and β remain.

From Fig.6 we can see that the stable area increases with increasing value of β, which means that

the more we consider the effects of the adjusting term, the more stable the traffic flow will be. In addition, the point A_1 means that if β=0.1, when the value of α is larger than the value of A_1, that is, 1.8, the traffic flow is always at the steady state. When α is lower than 1.8, the phase transition will occur from the free traffic, to the coexisting traffic, and then to the traffic jam. The meaning of points A_2=1.6 and A_3=1 are the same. This will be seen clearly with numerical simulations in the next section.

4. Numerical simulations for phase transitions

In this section the phase transitions are studied for the model (4) with numerical simulations. We consider the periodic boundary condition. The initial condition is given as follows:

$$x_1(0) = 0.1, \quad x_n(0) = -h(n-1), \quad n = 2, ..., N$$

and

$$\dot{x}_n(0) = 0 \quad \text{for} \quad n = 1, \ldots, N. \tag{11}$$

Total number of cars is N=100. The length of the road L is changed to get various h.

Dependences of the behaviour of headway for various h on the values of α and β are discussed in the following. Without any loss of generality, we let β=0.2 and 0.5 respectively.

If β=0.2, as shown in Fig.6, when the value of α is larger than A_2=1.6, the headway of traffic flow shows a homogeneous behaviour for any value of h and no jamming transition occurs. This also holds for $\alpha \geq A_3 = 1$, β=0.5. Figure 7 shows the plots of speed against headway for all cars at time steps 9000–10000 with α=1.6, β=0.2 in Fig.6(a) and α=1, β=0.5 in Fig.6(b). A single point in the plot represents the homogeneous traffic flow for an initial h. The dashed line is the optimal velocity function (2). The two plots of speed against headway both agree with the optimal velocity function (2).

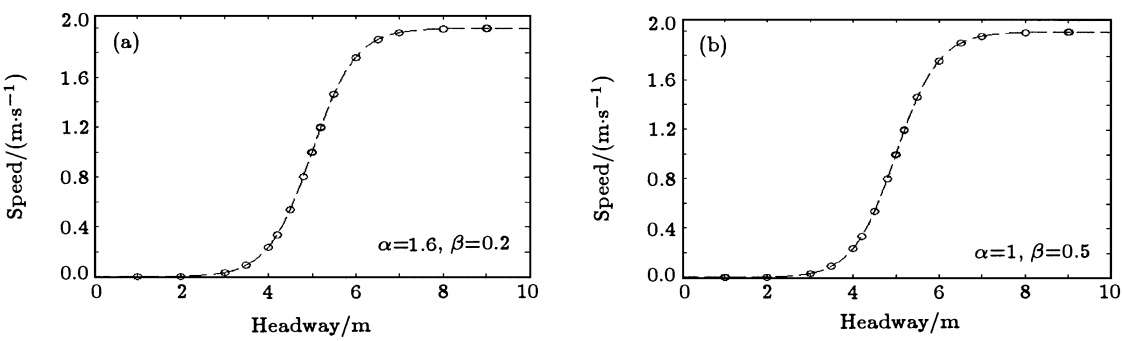


Fig.7. Plots of speed $\dot{x}_n$ against headway Δx_n for all cars at time steps 9000–10000 with α=1.6, β=0.2 in (a) and α=1, β=0.5 in (b).

If the value of α is less than 1.6 for β=0.2, the phase transition occurs among the free traffic, the coexisting traffic, and the traffic jam. Figure 8(a) shows the plots of the headway Δx_n against the car n for h=2, 5 and 7m and Fig.8(b) shows the plot of speed $\dot{x}_n$ against the headway Δx_n for all cars with α=1 at time step 10000. When the initial headway is around h=5, the inhomogeneous traffic appears and only two distinct headways (the higher is the free state and the lower is the jam state) stay. In Fig.8(b), the homogeneous traffic is represented by a single point and the inhomogeneous traffic is represented by a loop since the speed oscillates around the optimal speed with time development. The loop around h=5 indicates the unstable region of Fig.6 for β=0.2.

The analysis is similar for the value of α less than 1 for β=0.5. Figure 9(a) shows the plots of the headway Δx_n against the car n for h=2, 5 and 7m, and Fig.9(b) the plot of speed $\dot{x}_n$ against the headway Δx_n for all cars with α=0.6 at time step 10000.

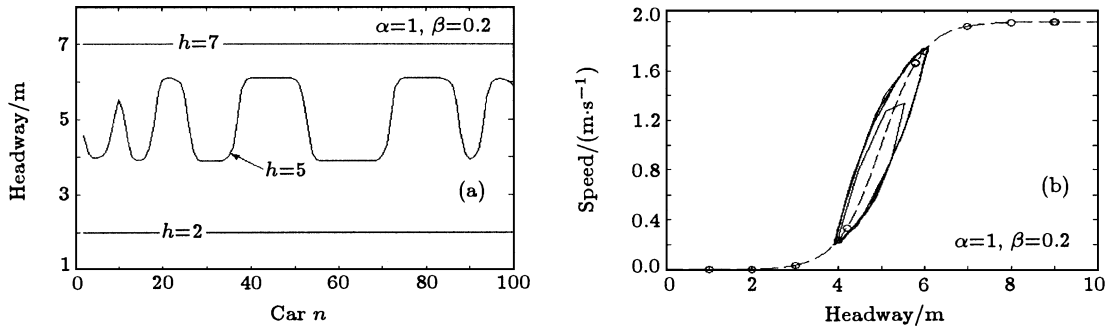


Fig.8. (a) Plots of the headway Δx_n against the car n for h=2, 5 and 7m and (b) plots of speed $\dot{x}_n$ against the headway Δx_n for all cars at time step 10000 with α=1, β=0.2. In Fig.9(b), the homogeneous traffic is represented by a single point and the inhomogeneous traffic is represented by a loop. The dashed line is the optimal velocity function (2).

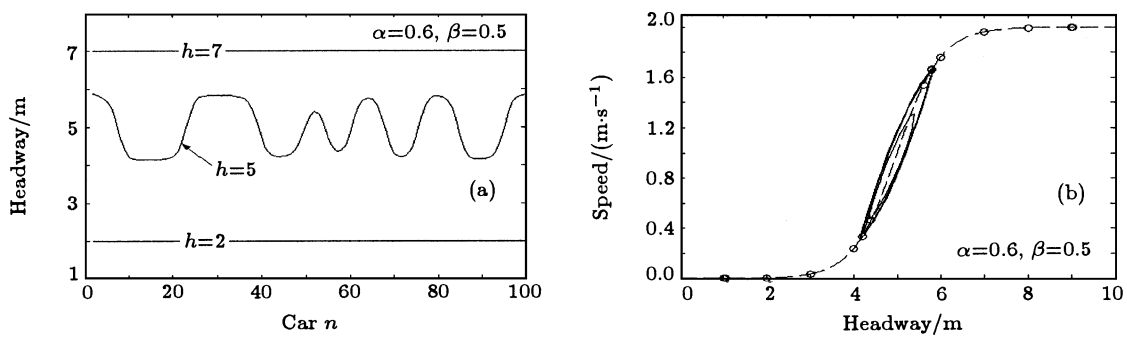


Fig.9. (a) Plots of the headway Δx_n against the car n for h=2, 5 and 7m and (b) plots of speed $\dot{x}_n$ against the headway Δx_n for all cars at time step 10000 with α=0.6, β=0.5. In Fig.9(b), the homogeneous traffic is represented by a single point and the inhomogeneous traffic is represented by a loop. The dashed line is the optimal velocity function (2).

Figure 10 shows the plots of the headway Δx_n against the car n for h=2, 5 and 7m with α=1 and 0.5 for β=0.2 in Fig.10(a) and α=0.5 for β=0.2 and 0.5 in Fig.10(b) at time step 10000. From Fig.10(a) we can see that for a given value of β, the amplitude of the oscillating inhomogeneous headway with larger α is smaller than that with smaller α. On the other hand, for a given value of α, the amplitude of the oscillating inhomogeneous headway with larger β is smaller than that with smaller β.

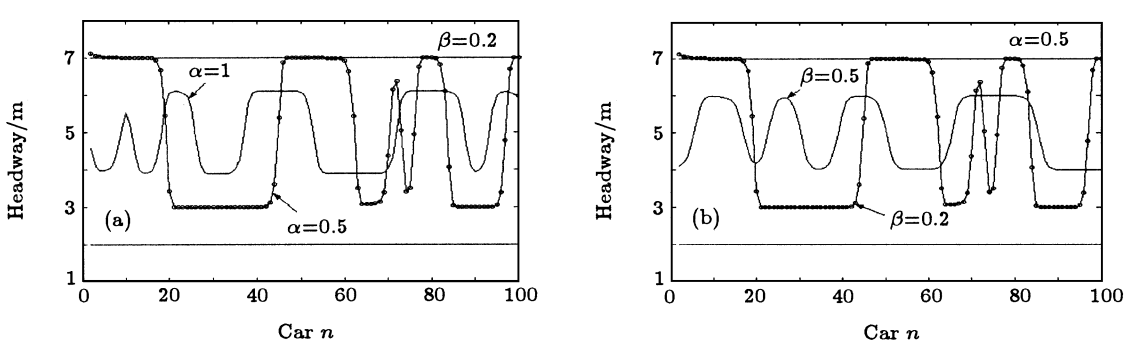


Fig.10. Plots of the headway Δx_n against the car n for h=2, 5 and 7$m$with α=1 and 0.5 for β=0.2 in (a) and α=0.5 for β=0.2 and 0.5 in (b) at time step 10000.

We plot the data in the space of speed $\dot{x}_n$ against headway Δx_n with α=0.4 and 1.2 for β =0.2 and α=0.5 for β =0.2 and 0.5 in Fig.11. Let h=5 in both situations. Because the speed oscillates around the optimal speed with time development, the inhomogeneous traffic is represented by a loop, which the upper cusp point denotes the free traffic and the lower one denotes the jam traffic. For a given value of β, the hysteresis loop with larger α is smaller than that with smaller α. And for a given value of α, the hysteresis loop with larger β is smaller than that with smaller β.

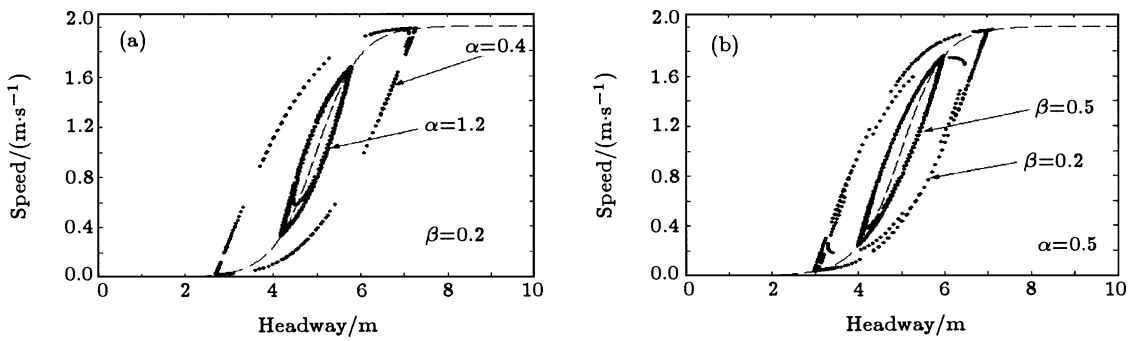


Fig.11. Hysteresis loops in the space of speed $\dot{x}_n$ against headway Δx_n with α=0.4, 1.2, β =0.2 (a) and α=0.5, β =0.2, 0.5 (b) for h=5. The dashed lines represent the optimal velocity function (2).

Figures 10(a) and 11(a) indicate that with decreasing value of α, the oscillation of the inhomogeneous headway increases. At the same time, the Figs.10(b) and 11(b) indicate that with increasing value of β, the oscillation of the inhomogeneous headway decreases.

5. Summary

A new car-following model performing more realistically in both acceleration process and deceleration process is proposed in this paper. We study this new model with the linear stability method. If the initial homogeneous headway of the traffic flow does not satisfy the linear stable condition (9) determined by the values of α and β, the homogeneous traffic flow is unstable and the coexisting flows with both free and jammed states appear. Phase transitions among free traffic flow, coexisting flow and traffic jams occur, as shown in Figs.8 and 9. For coexisting phase, the kink–antikink oscillating waves appear. Numerical simulations show that the amplitude of the oscillating inhomogeneous headway increases with decreasing value of α or β. This means that for a very small value of α (correspond to a realistic value of relaxation time $T_{\rm r}$), the stability of the traffic flow can be strengthened by choosing an appropriate larger value of β.

References

[1] Douglas A Kurtze 1995 *Phys. Rev.* E **52** 218
[2] Takashi Nagatani 1998 *Phys. Rev.* E **58** 4271
[3] Kerner B S and Rehborn H 1997 *Phys. Rev.* E **79** 4030
[4] Xue Y 2002 *Chin. Phys.* **11** 1128
[5] Jiang R and Wu Q S 2003 *Physica* A **322** 676
[6] Bando M, Hasebe K, Nakayama A, Shibata A and Sugiyama Y 1995 *Phys. Rev.* E **51** 1035
[7] Takashi Nagatani 2000 *Phys. Rev.* E **61** 3534
[8] Treiber M, Henneche A and Helbing D 2000 *Phys. Rev.* E **62** 1805