Vol 14 No 3, March 2005
1009-1963/2005/14(03)/0571-05

Chinese Physics

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Propagation and collision of compacton-like kinks in Klein–Gordon lattice system*

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(Received 4 August 2004; revised manuscript received 27 September 2004)

We study the propagation and collision of the compacton-like kinks in the system of an anharmonic lattice with a double well on-site potential by a direct algebraic method and numerical experiments. It is found that the localization of the compacton-like kinks is related to the nonlinear coupling parameter C_{nl} and the potential barrier height V_0 of the double well potential. The velocity of the propagation of the compacton-like kinks is determined by the linear coupling parameter C_l, the nonlinear coupling parameter C_{nl} and the localization parameter q. Numerical experiments demonstrate that appropriate C_l is not detrimental to a stable propagation of the compacton-like kinks. However, the collision of compacton-like kinks and anti-kinks in the lattice with comparatively small C_l leads to the emergence of a discrete stationary breather and small amplitude nonlinear oscillation background, while moderate C_l results in the emergence of two deformed kinks with radiating oscillations and lower propagation velocities.

Keywords: Klein–Gordon lattice, compacton-like kink, propagation, collision
PACC: 6320P, 6320R

1. Introduction

In recent years, the study of nonlinear localized modes in perfect nonlinear lattice systems has received a great deal of attention in condensed matter physics[1] since the pioneering work of Sievers and Takeno.[2] Many models (one-dimensional monatomic lattice and diatomic lattice) and many approaches have been emphasized in connection with the soliton-like envelope solutions.[3−9] This type of wave with exceptional properties however presents some wings or tails at infinity, while the observed patterns in nature, whether stationary or travelling, are of finite extent. Indeed, strictly localized solitary waves without infinite tails, i.e. compactons, has appeared recently in the literature.[10−27] Since Rosenau and Hyman[10,11] investigated a special type of Korteweg–de Vries equation, and found that solitary waves may compactify in the presence of a nonlinear dispersion, many authors have studied the compacton-like solutions in the related physics models,[12−27] especially in the discrete lattice models with nonlinear coupling and on-site potential, which allows the propagation of compacton-like kinks and pulses.[17,19,20,25−27] Kivshar[16] has reported that intrinsic localized modes in purely anharmonic lattices may exhibit compacton-like properties. Recently, Dusuel *et al*[17] have demonstrated that the same phenomenon can also appear in a nonlinear Klein–Gordon lattice with anharmonic coupling, and obtained the experimental evidence of the existence of a static compacton in a real system, made up by identical pendulums connected by a spring. In addition, there exists a moving compacton-like kink with a single velocity $v = \sqrt{C_l}$ in a continuous Klein–Gordon system with on-site potential, implying that the presence of harmonic coupling forces between the lattice sites is required for a compacton-like kink to propagate. Very recently, Dinda and Remoissenet reported their work in Ref.[18]. They found that there are breather compactons in a nonlinear Klein–Gordon lattice with a soft on-site potential; in Ref.[19], they

*Project supported by the Nature Science Foundation of Hunan Province (Grant No 02JJY2079) and the Youth Foundation of School of Physics Science and Technology of Central South University.

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demonstrated that the effects of lattice discreteness, and the presence of a linear coupling between lattice sites, are detrimental to a stable ballistic propagation of the compacton. However, in Refs.[17–19] the continuum limit approximation and some other approximations have been used.

In this paper, we reconsider the lattice model of Refs.[17, 19], where a system of atoms with unit mass are coupled harmonically or anharmonically to their nearest-neighbours and interact with a double well on-site potential. Different from the previous studies,[17,19] we use not any approximation but a direct algebraic method. An analytical solution is chosen as an initial condition of the system to check the stability of the moving compacton-like kink solution and the collisions of the compacton-like kink and anti-kink in our numerical experiments. The paper is organized as follows. We present the lattice model and give its analytical compacton-like kink solution with certain velocity in Section 2. Then, in Section 3, we study numerically the propagation of compacton-like kinks and the collisions of compacton-like kinks and anti-kinks. Conclusions are drawn in Section 4.

2. Model and formalism

We consider the nonlinear Klein–Gordon lattice system; its Hamiltonian is given by[17,19]

$$H = \sum_n \left[\frac{1}{2} \left(\frac{\mathrm{d}Q_n}{\mathrm{d}t} \right)^2 + \frac{1}{2} C_l \left(Q_n - Q_{n-1} \right)^2 + \frac{1}{4} C_{nl} \left(Q_n - Q_{n-1} \right)^4 + \frac{1}{2} V_0 \left(1 - Q_n^2 \right)^2 \right], \quad (1)$$

where Q_n is the longitudinal displacement of the nth particle from its equilibrium position, C_l, C_{nl} and V_0 are parameters that control the strength of the linear coupling, nonlinear coupling between the nearest-neighbour lattice site and the potential barrier height of the double well potential, respectively. And $(1 - Q_n^2)^2$ is the symmetric on-site potential of the system. The equation of motion for the system is

$$\frac{\mathrm{d}^2 Q_n}{\mathrm{d}t^2} = C_l \left(Q_{n+1} + Q_{n-1} - 2Q_n \right) + C_{nl} \left[\left(Q_{n+1} - Q_n \right)^3 + \left(Q_{n-1} - Q_n \right)^3 \right] + 2V_0 \left(Q_n - Q_n^3 \right). \quad (2)$$

To find a moving localized solution of equation (2), we use the fundamental idea proposed by Rosenau and Hyman,[10] assuming that such a moving localized solution may be taken as a part (one half of the periodic) of a quasi-linear periodic solution with finite wavelength. We suppose the equation to have moving kink solution with compact support, and seek a solution of the form

$$Q_n(t) = \sin \left[q \left(na - n_0 a \pm vt \right) \right]$$

for

$$\left| q \left(na - n_0 a \pm vt \right) \right| < \pi/2, \quad (3a)$$

$$Q_n = -1$$

for

$$q \left(na - n_0 a \pm vt \right) \leq -\pi/2, \quad (3b)$$

$$Q_n = 1$$

for

$$q \left(na - n_0 a \pm vt \right) \geq \pi/2, \quad (3c)$$

where a is the lattice spacing and is assumed equal to unity in what follows, n_0 is the centre site of the compacton-like kink, sign "+" and "–" correspond to left-travelling and right-travelling compacton-like kinks, respectively, and q is a positive parameter ($0 < q < \pi/2$) indicating the localization of the compact kink, v is the propagation velocity of compact kink. Both q and v will be determined later. The invariance of Eq.(2) under the transformation $Q_n \to -Q_n$, $t \to -t$ permits negative anticompactons propagating in the opposite direction. The full width of the compacton-like kink is $L_c = \pi/q$, which implies that a compacton and an anticompacton will not interact unless they come into contact in a way similar to the contact between two hard spheres. Substituting Eq.(3a) into Eq.(2), and equating the coefficient of $\sin \left[q \left(n - n_0 \pm vt \right) \right]$ and $\sin^3 \left[q \left(n - n_0 \pm vt \right) \right]$ on both sides of Eq.(2) respectively, we obtain

$$\left(1 + 2\cos q \right) \left(1 - \cos q \right)^2 = \frac{V_0}{2C_{nl}}, \quad (4)$$

$$\left(qv \right)^2 = 2C_l (1 - \cos q) + 2C_{nl} (1 - \cos q)^3. \quad (5)$$

Equation (4) indicates that the localization (which is determined by the value of q, the larger the value of q, the more localized the compacton) of the compact kink is related to the nonlinear coupling parameter C_{nl} and the potential barrier height V_0 of the double well potential. Combining Eqs.(4) and (5), we have the relation

$$v = \sqrt{2C_l (1 - \cos q) + V_0 (1 - \cos q)/(1 + 2\cos q)}/q, \quad (6)$$

which indicates that the propagating speed of the compacton-like kink is determined by the linear coupling C_l, the localization parameter q and the potential barrier height V_0. If $C_l = 0$, v is not zero. This result is different from the previous works.[17,19] In Refs.[17] Dusual *et al* have demonstrated that there exists moving compacton-like kink with a single velocity $v = \sqrt{C_l}$ in a continuous Klein–Gordon system with on-site potential, implying that the presence of harmonic coupling forces between the lattice sites is required for a compacton-like kink to propagate; if $C_l = 0$, only static compacton exists. In Ref.[19], the authors have indicated that a compacton-like kink can execute a stable ballistic motion at any non-relativistic velocity.

3. Numerical experiments

In order to investigate the effects of linear coupling on the propagation and collision of compacton-like kinks in the system, we have numerically integrated Eq.(2) by means of fourth-order Runge–Kutta scheme with a time step chosen to preserve the total energy of the system to a good accuracy over a complete run.

The analytical solution stated above is chosen as an initial condition of the system to check the stability of the moving compacton-like kink solution of this lattice system. To compare with Refs.[17, 19], in this paper we take the potential barrier height $V_0 = 2$, the full width of the compacton-like kink L_c=50 ($q = \pi/50$) with n=1000, and L_c=500 ($q = \pi/500$) with $n = 10000$, respectively.

We first consider the propagation of the compacton-like kinks. At the initial time $t = 0$, both of the compacton-like kinks are centred on the centre of the chain, i.e. n_0=500 and n_0=5000 respectively. We systemically study the effect of the linear parameter C_l on the propagation of compacton-like kinks by changing the value of C_l step by step. We find that, for appropriate C_l, the compacton-like kinks can travel along the chain freely without emission of radiation. For the compacton-like kink with L_c=50 ($q = \pi/50$) and n=1000, we take the velocity v=15 ($C_l \sim 225$), and for L_c=500 ($q = \pi/500$) with n=10000, we take the velocity $v = 100$ ($C_l \sim 10000$) as examples, respectively. Our numerical results are shown in Figs.1 and 2. After travelling on the lattice chain for time t=30 seconds (L_c=50) and t=40 seconds (L_c=500) respectively, the compacton-like kinks keep their profile and propagate along the chain without emission of radiation or with the radiation being below the numerical accuracy, and the propagation velocities of the compacton-like kinks remain unchanged, which shows that the moving compacton-like kinks obtained in this lattice system with appropriate C_l is stable and hence our method is valid. In the previous studies,[17,19] the continuum limit approximation has been used, although this approximation is valid when discreteness effects are negligible, such an initial condition acts as a source of perturbation and emits small radiation. This radiative process alters the compact support of the compacton-like kink and causes a small decrease of its velocity.

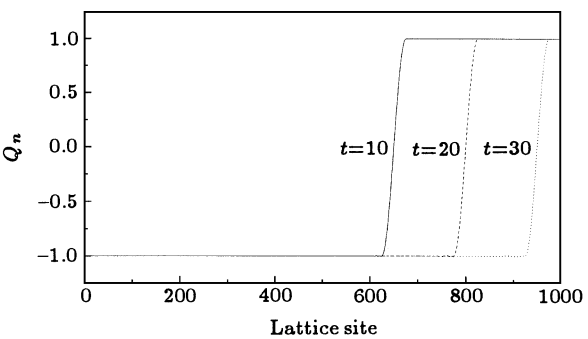


Fig.1. Profiles of a compacton-like kink after travelling on a lattice chain (n=1000) for t=10s, t=20s and t=30s respectively with the corresponding localization parameter $q = \pi/50$ and velocity v=15.

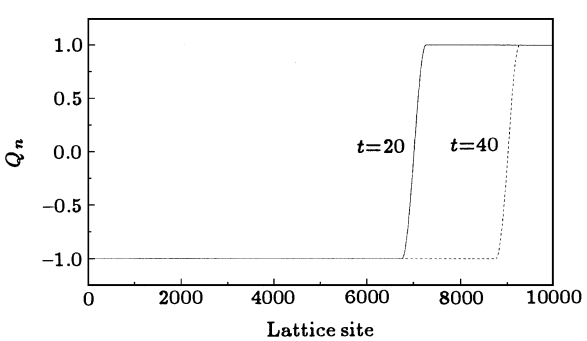


Fig.2. Profiles of a compacton-like kink after travelling on a lattice chain (n=10000) for t=20s and t=40s respectively with the corresponding localization parameter $q = \pi/500$ and velocity v=100.

Next, we display the head-on collision results of the compacton-like kinks and anti-kinks. At the initial time t=0, the compacton-like kinks and anti-kinks are away from $20a$ and $200a$ for the chain n=1000 and n=10000, respectively. For comparatively small C_l (we take v=5, i.e. $C_l \sim 25$ for compactons with L_c=50 and n=1000, and v=25, i.e. $C_l \sim 2500$ for compactons with L_c=500 and n=10000), a discrete

stationary breather and small amplitude nonlinear oscillation background emerge from the weakly inelastic compacton-like kink and anti-kink collision as shown in Figs.3 and 4: this discrete breather has a significant lifetime and presents a genuine physical interest. Indeed, their collision may be compared to that of two hard spheres, i.e. without long-distance interaction.

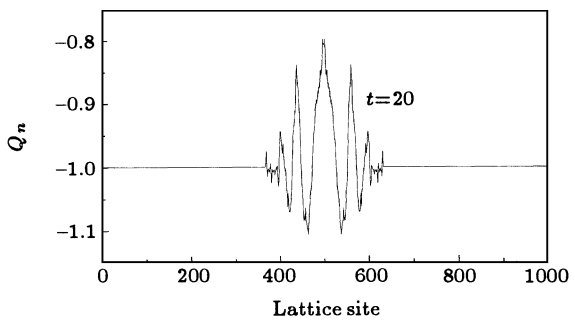


Fig.3. Plot showing the emergence of a discrete stationary breather and nonlinear oscillation background after collision of a compacton-like kink with the corresponding localization parameter $q = \pi/50$, velocity v=5 and an anti-kink with velocity v=–5 at t=20s.

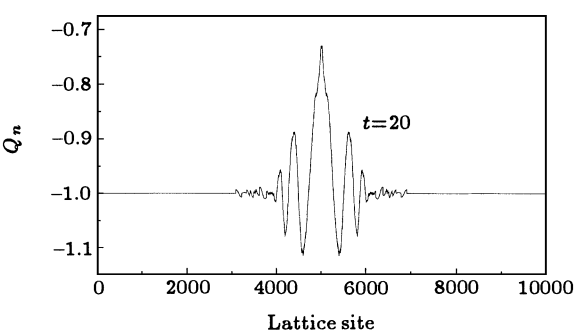


Fig.4. Plot showing the emergence of a discrete stationary breather and nonlinear oscillation background after collision of a compacton-like kink with the corresponding localization parameter $q = \pi/500$, velocity v=50 and an anti-kink with velocity v=–50 at t=20s.

For moderate C_l, the head-on collision of the compacton-like kinks and anti-kinks is inelastic: the two kinks that emerge from the collision are deformed; they radiate oscillations and propagate at velocities lower than the initial velocities. For the compacton-like kink and anti-kink with L_c=50 ($q = \pi/50$) and n=1000, we take the velocity $v = 9$ ($C_l \sim 81$) and $v = 15$ ($C_l \sim 225$). After the collision of the compactons, the deformed kinks propagate with radiate oscillations at velocities $v \sim 6.4$ and $v \sim 14$ respectively, the process of the collisions are presented in Figs.5 and 6.

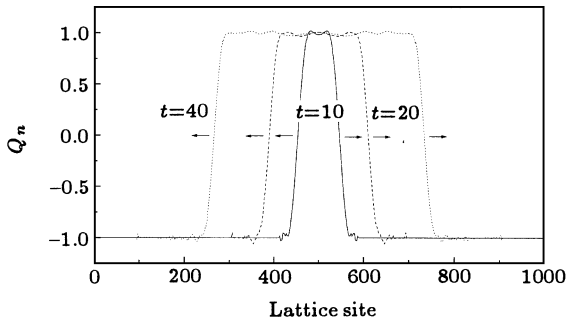


Fig.5. Plot showing the collision process of a compacton-like kink with the corresponding localization parameter $q = \pi/50$, velocity v=9 and an anti-kink with velocity v=–9 at t=10s, t=20s and t=40s.

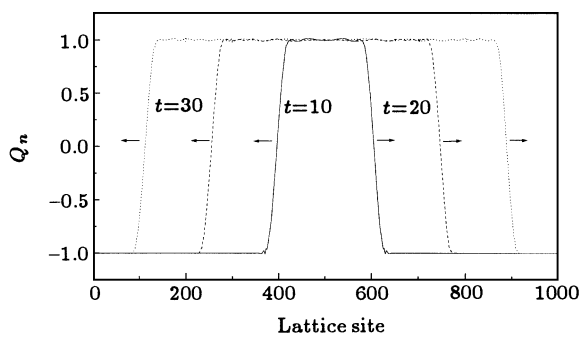


Fig.6. Plot showing the collision process of a compacton-like kink with the corresponding localization parameter $q = \pi/50$, velocity v=15 and an anti-kink with velocity v=–15 at t=10s, t=20s and t=30s.

The slow kinks' velocities change almost 30% from the initial compactons' velocities, while the speedy kinks' change is less than 7%. The significant difference comes from the fact that the slow kinks radiate many phonons in the whole background during the propagation while the speedy ones only radiate phonons in the collision place. For L_c=500 ($q = \pi/500$) with n=10000, the collisions of the compactons have the same results; we only take the velocity v=100 ($C_l \sim 10000$) as an example for the purpose of saving place. After the collision of the compactons, the deformed kinks propagation with radiate oscillations in the background at velocity $v \sim 78.5$ as shown in Fig.7. However, in Ref.[19], the authors demonstrate the results of the head-on collision of the large compactons only with comparatively small velocity, and the collision process ends in an unstable bound state.

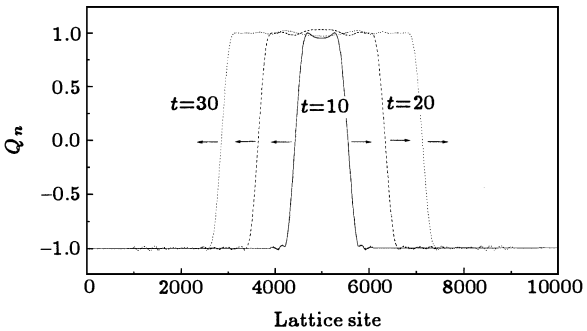


Fig.7. Plot showing the collision process of a compacton-like kink with the corresponding localization parameter $q = \pi/500$, velocity v=100 and an anti-kink with velocity v=−100 at t=10s, t=20s and t=30s respectively.

4. Conclusion

In summary, we have studied the propagation and collision of compacton-like kinks in the system of an anharmonic lattice with symmetric double well on-site potential by a direct algebraic method and numerical experiments. It is found that the localization of the compacton-like kinks is related to the nonlinear coupling parameter C_{nl} and the potential barrier height V_0 of the double well potential, and the velocity of the propagation of the compacton-like kinks is determined by the linear coupling parameter C_l, the nonlinear coupling parameter C_{nl} and the localization parameter q. Numerical experiments demonstrate that appropriate C_l is not detrimental to a stable propagation of the compacton-like kinks; however, the collisions of compacton-like kinks and anti-kinks in the chain with comparatively small C_l lead to the emergence of a discrete stationary breather and small amplitude nonlinear oscillation background, while moderate C_l results in the emergence of two deformed kinks with radiating oscillations and lower propagation velocities. This result is somewhat different from the previous works[17,19] since the continuum limit approximation and some other approximations have been used in these references. Although this approximation is valid when discreteness effects are negligible, such an initial condition acts as a source of perturbation and causes a small decrease of its velocity. It emerges from previous related studies[10−27] as well as this paper that the concept of compacton implies strict conditions that must be satisfied in order for a physical system to support solitary waves with compact support; and much attention must be paid to make the compacton concept a reality in practical applications such as signal processing and fiber communication.[26]

References

[1] Flach S and Willis C R 1998 *Phys. Rep.* **295** 181
[2] Sievers A J and Takeno S 1988 *Phys. Rev. Lett.* **61** 970
[3] Page J B 1990 *Phys. Rev.* B **41** 7835
[4] Huang G X and Hu B B 1998 *Phys. Rev.* B **57** 5746
Huang G X 2001 *Chin. Phys.* **10** 523
[5] Franchini A, Bortolani V and Wallis R F 1998 *Phys. Rev.* B **58** 8391
[6] Zhou G H, Xia Q L and Yan J R 2000 *Acta Phys. Sin.* **49** 1741 (in Chinese)
[7] Zhou G H, Xia Q L and Yan J R 2000 *Eur. Phys. J.* B **17** 207
Zhou G H, Xia Q L and Yan J R 2001 *Eur. Phys. J.* B **24** 297
[8] Zhou G H, Xia Q L, Pan L X and Yan J R 2000 *Chin. Phys. Lett.* **17** 899
[9] Wang D L, Yan X H and Tang Y 2003 *Chin. Phys. Lett.* **20** 554
[10] Rosenau Ph and Hyman J M 1993 *Phys. Rev. Lett.* **70** 564
[11] Rosenau Ph 1994 *Phys. Rev. Lett.* **73** 1737
[12] Cooper F, Shepard H and Sodano P 1993 *Phys. Rev.* E **48** 4027
[13] Rosenau Ph 1996 *Phys. Lett.* A **211** 265
[14] Dey B 1998 *Phys. Rev.* E **57** 4733
[15] Rosenau Ph 2000 *Phys. Lett.* A **275** 193
[16] Kivshar Yuri S 1993 *Phys. Rev.* E **48** 43
[17] Dusual S, Michaux P and Remoissenet M 1998 *Phys. Rev.* E **57** 2320
[18] Dinda P T and Remoissenet M 1999 *Phys. Rev.* E **60** 6218
[19] Dinda P T, Kofane T C and Remoissenet M 1999 *Phys. Rev.* E **60** 7525
[20] Comte J C 2002 *Phys. Rev.* E **65** 046619
Comte J C 2002 *Phys. Rev.* E **65** 067601
[21] Dey B, Eleftheriou M, Flach S and Tsironis G P 2002 *Phys. Rev.* E **65** 017601
[22] Eleftheriou M, Dey B and Tsironis G P 2000 *Phys. Rev.* E **62** 7540
[23] Kevrekidis P G and Konotop V V 2002 *Phys. Rev.* E **65** 066614
[24] Kevrekidis P G, Konotop V V, Bishop A R and Takeno S 2002 *J. Phys.* A: *Math. Gen.* **35** L641
[25] Xia Q L and Jiang D C 2003 *Chin. Phys. Lett.* **20** 105
[26] Comte J C 2003 *Int. J. Bifurc. Chaos* **13** 1565
[27] Xia Q L and Jiang D C 2004 *Commun. Theor. Phys.* **42** 295