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# An active control synchronization for two modified Chua circuits*

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From modern control theory, an active control method to synchronize two modified Chua circuits with each other, which exhibit chaos, is presented. Some sufficient conditions of linear stability of the chaotic synchronization are obtained from rigorous mathematic justification. On the basis of the state-observer, the controller is analytically deduced using the active control. It is shown that this technique can be applied to achieve synchronization of the two systems with each other, whether they are identical or not. Finally, numerical simulations show the effectiveness of the proposed control scheme.

**Keywords:** chaos, synchronization, modified Chua circuit, active control, state observer
**PACC:** 0545

## 1. Introduction

Chaos, as a very interesting nonlinear phenomenon, has been intensively studied in the last three decades after the pioneering work by Ott *et al.*[1] Several methods for control and stabilization of chaotic motions have recently been presented.[1−4] Over the last decade, chaos synchronization has aroused considerable interests in many areas of science and technology due to its potential application to secure communications.[5]

Chua circuit has been studied extensively as a prototypical electronic system. In Ref.[6], the adaptive backstepping control has been applied to several kinds of Chua circuits with no parameter known. Recently, the control of the modified Chua circuit system was put forward by Hartley and Mossayebi[7] by designing a controller for tracking the capacitor voltage of the system with the input–output and state space techniques. Some investigators, such as, Saito and Mitsubori,[8] and Hwang *et al*[9] stabilized a modified Chua chaotic attractor to a desired orbit.[8] With the scheme originally suggested in Ref.[10], the present study is to apply the active control to synchronize two modified Chua circuits with each other.

This paper is outlined as follows. The chaotic response of the modified Chua system is detailed in Section 2. In Section 3, the synchronizing mechanism of the method is developed analytically. The active control is applied to two modified Chua systems and numerical simulations are performed to testify its availability. Finally, in Section 4, some concluding remarks are made.

## 2. A modified Chua circuit

Hwang *et al*[9] showed that the piecewise nonlinearity of the Chua circuit could be replaced by a cubic nonlinearity with little change in the system dynamics or bifurcation structure. The modified Chua circuit is described by a set of ordinary differential equations[9]

$$\begin{cases} \dot{x} = -py, \\ \dot{y} = x - y + z, \\ \dot{z} = q(y - \dfrac{1}{7}(2z^3 - z)), \end{cases} \tag{1}$$

where the parameters $p = 100/7$ and $q = 10$ are chosen to ensure the chaotic behaviour of the system (1), $y$ and $z$ are the voltages across two capacitors respectively, and $x$ is the current through the inductor. For these parameters, numerical computations indicate that the Lyapunov exponents for the system (1) are $[0.49,\ 0.0,\ -4.6]$. From the analysis of equilibrium point, the system (1) is deduced to have three equilibrium points of $E_+ = (\sqrt{0.5},\ 0,\ -\sqrt{0.5})$, $E_0 = (0,\ 0,\ 0)$

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and $E_- = (-\sqrt{0.5},\ 0,\ \sqrt{0.5})$. By analyzing the Jacobian matrix of the system (1) about the three equilibrium points accompanied by Routh–Hurwitz criteria, Ref. [11] indicates that the equilibrium points are unstable. Figure 1 indicates the chaotic attractor of the system (1) in the $y$–$z$ plane.

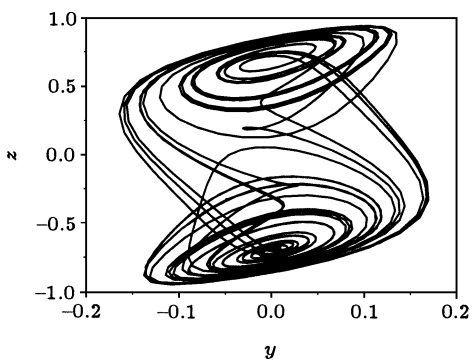



**Fig.1.** Chaotic attractor of the modified Chua circuit in the $y$–$z$ plane.

**Remark 1** Like many other chaotic systems, a modified Chua circuit is a bounded, zero-volume, global set. Therefore, for $t \geq 0$, the state trajectories $x(t)$, $y(t)$ and $z(t)$ are all globally bounded and continuously differentiable with respect to time. Consequently, $\forall\ t \geq 0$, $\exists\ s$ and $0 < s < \infty$, so that $|x(t)| < s$,$|y(t)| < s$ and $|z(t)| < s$.

Hartley and Mossayebi[7] designed a controller for tracking the capacitor voltage in a modified Chua circuit system by applying the input–output and state space techniques. With a simple control method, Saito and Mitsubori [8] forced the chaotic attractor to a desired orbit. Hwang *et al*[9] proposed a controller including two portions of feedback to suppress chaotic states of a modified Chua circuit system. Compared with the above methods, the active control method is adopted to synchronize two modified Chua circuits with each other in the next section.

# 3. Synchronization of two modified Chua circuits with each other using active control

## 3.1. Active control method

Before synchronizing two chaotic systems with each other, we mention the active control method in this section. Let us consider two $n$-dimensional autonomous dynamical systems described by the differential equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \tag{2a}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}), \tag{2b}$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are all the $n$-dimensional vectors, $\boldsymbol{f} = (f1, \ldots, f_n)$ is a nonlinear vector-valued function, and the dot over $\boldsymbol{x}$ (or $\boldsymbol{y}$) denotes the first derivative of $\boldsymbol{x}$ (or $\boldsymbol{y}$) with respect to time. The dynamical system of Eq.(2a) is called the driver, master, or sender system, while the system of Eq.(2b) is called the response, slave, or receiver system in the literature. To achieve synchronization, we transfer Eq.(2b) to

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) \tag{2c}$$

by adding the controller $\boldsymbol{u}$, where $\boldsymbol{u} = (u_1, \ldots, u_n) \in R^n$. Synchronization between the driver and response systems is said to be achieved if $\|\boldsymbol{e}(t)\| \to 0$ and $\boldsymbol{u} \to 0$ as $t \to \infty$ by introducing synchronization error of $\boldsymbol{y}(t)$ from $\boldsymbol{x}(t)$ as

$$\boldsymbol{e}(t) = \boldsymbol{y}(t) - \boldsymbol{x}(t). \tag{3}$$

Note that the so-called synchronization here refers to complete synchronization. From Eqs.(2a), (2c), and (3), it is known that

$$\begin{aligned}\dot{\boldsymbol{e}}(t) =& \dot{\boldsymbol{y}}(t) - \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{e}(t)) \\ &+ \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}(t)) - \boldsymbol{f}(\boldsymbol{x}).\end{aligned} \tag{4}$$

Generally, in order that the synchronization is achieved, one has to find such a controller $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ that Eq.(4) has a stable fixed point at $\boldsymbol{e}(t) = 0$. Thus, the synchronization problem can be seen as the stabilization of the system (4) at the origin via an adequate $\boldsymbol{u}$. $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ may be in various forms consisting of linear or nonlinear combinations. In this paper, a general form of the feedback function $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ can be given in a systematic way. Following the example in Ref.[9], one can take the feedback function as follows:

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{L}(\boldsymbol{e}) + \boldsymbol{N}(\boldsymbol{e}), \tag{5}$$

where $\boldsymbol{L}(\boldsymbol{e})$ and $\boldsymbol{N}(\boldsymbol{e})$ are the linear and nonlinear functions of $\boldsymbol{e}$, respectively. There are several possible choices for $\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y})$ so that there will exist different approaches to zero as time increases. One possible set for ensuring Eq.(4) is the following:

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{C}e(t), \tag{6}$$

where $\boldsymbol{C} \in R^{n \times n}$ is a constant matrix. $\boldsymbol{e}(t)$ will converge to zero if all the eigenvalues of $\boldsymbol{C}$, if any, have

negative real parts. We note that if $\boldsymbol{C}$ is of nilpotent, dead-beat synchronization will be achieved.

### 3.2. Numerical simulations for two modified Chua circuits

To illustrate the proposed approach, we take the modified Chua circuit represented by Eq.(1) as an application in this section. Suppose that we have two modified Chua Circuits, i.e. the drive system with the subscript 1 and the response system with subscript 2. Our goal is to determine the controller $\boldsymbol{u}$. Subtracting the drive system from the driven system yields

$$\begin{aligned} e_1 &= -pe_2 + u_1, \\ e_2 &= e_1 - e_2 + e_3 + u_2, \\ e_3 &= q\left(e_2 - \frac{1}{7}(2z_2^3 - 2z_1^3 - e_3)\right) + u_3. \end{aligned} \tag{7}$$

The synchronization problem for the modified Chua circuit system is now converted to the achievement of the asymptotic stability of the zero solution of the system (7). We define the active controller as follows:

$$\begin{aligned} u_1 &= L_1, \\ u_2 &= L_2, \\ u_3 &= \frac{q}{7}(2z_2^3 - 2z_1^3) + L_3, \end{aligned} \tag{8}$$

which indicates that Eq.(7) can be expressed by the following dynamical system:

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} + \boldsymbol{L})\boldsymbol{e}, \tag{9}$$

where $\boldsymbol{L} \in R^{3\times 3}$ is a constant matrix, $\boldsymbol{A} = \begin{bmatrix} 0 & -p & 0 \\ 1 & -1 & 1 \\ 0 & q & q/7 \end{bmatrix}$. The proper choice of the elements in the matrix $\boldsymbol{L}$ is such that the Jacobian matrix of system (9) must have the eigenvalues all with negative real parts. Letting $\boldsymbol{L} = -\boldsymbol{B}^{\mathrm{T}}\boldsymbol{K}$, where $\boldsymbol{B} \in R^3$ and $\boldsymbol{K} \in R^3$ are constant matrices, the matrix $\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K}$ must have $\mathrm{Re}(\lambda_i(\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K})) < 0$ by selecting $\boldsymbol{K}$. It is well known that the solution exists if all the uncontrollable eigenvalues of $(\boldsymbol{A},\ \boldsymbol{B}^{\mathrm{T}})$, if any, have negative real parts and can be easily derived.

**Theorem 1**[12] Consider the system given by Eq.(9). If the pair $(\boldsymbol{A}, \boldsymbol{B}^{\mathrm{T}})$ is observable then there exists a constant gain matrix $\boldsymbol{K}$ so that $(\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K})$ is stable.

**Remark 2** The proposed technique similar to an observer herein generates a linear and time-invariant error system. Synchronization can be achieved only if the eigenvalues of $\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K}$ have negative real part.

The fourth-order Runge–Kutta method is used in all numerical simulations with time step being equal to 0.001. The initial conditions for the driving Chua circuit system and the response system are arbitrarily given by (0, 0, 0.65) and (−1.2, 0.2, −0.5), respectively. Let $\boldsymbol{B} = [0,\ -1,\ -1]$, $\lambda_i(\boldsymbol{A} - \boldsymbol{B}^{\mathrm{T}}\boldsymbol{K}) = [-5,\ -3\ -\ 2]$. According to Matlab, $\boldsymbol{K} = [-0.175,\ -6.475,\ -1.0964]$ is obtained, so $\boldsymbol{L} = \begin{bmatrix} 0 & 0 & 0 \\ -0.175 & -0.6475 & -1.0964 \\ -0.175 & -6.475 & -1.0964 \end{bmatrix}$, and the control functions $\boldsymbol{u}$ can be determined.

The simulation results are shown in Fig.2. Note that the control is not imposed on the slave system until time $t = 5$ seconds. Figure 2(a) displays the $x_1$ and $x_2$, Fig.2(b) displays the $y_1$ and $y_2$, and Fig.2(c) displays the $z_1$ and $z_2$. Figures 2(a)–2(c) display the same sequence of signals with the active control. We observe in numerical simulations that the synchronization between derive and response of two chaotic systems occurs after applying the active control at $t = 2$s. The evolution of synchronization error is shown in Fig.2(d). As can be seen from Fig.2, after several seconds there exists synchronization between two chaotic systems starting from different initial conditions. It could be deduced that the synchronization occurs even though they have different structures. The simulation is not shown here due to the length limitation of this paper.

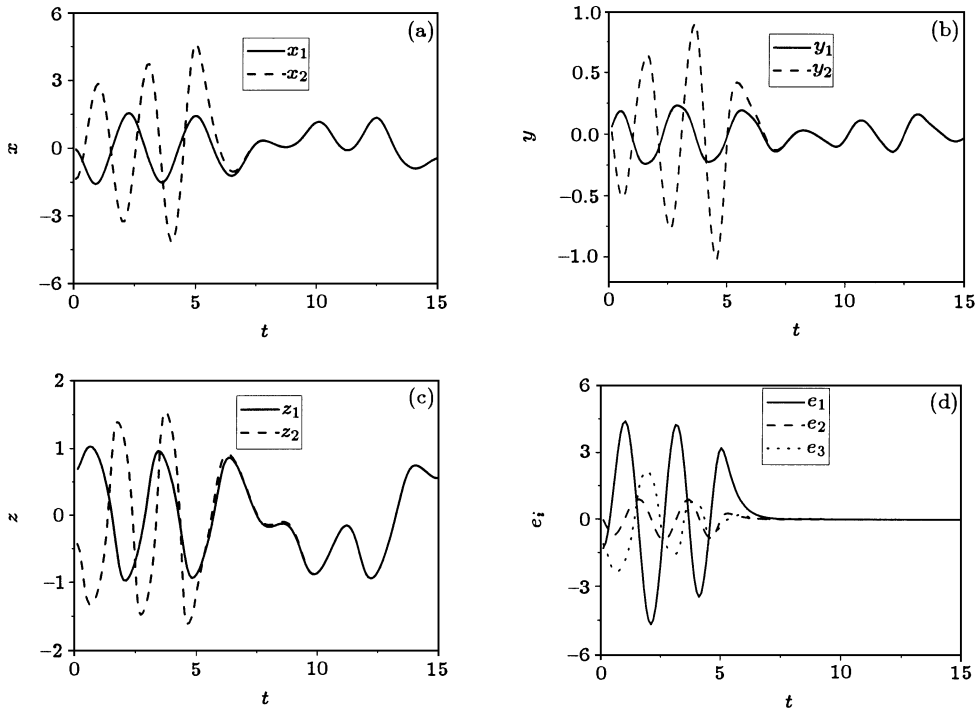



**Fig.2.** Synchronization of two modified Chua circuits with each other by use of the active control (a) signals $x_1$ and $x_2$, (b) signals $y_1$ and $y_2$, (c) signals $z_1$ and $z_2$. ($x_1$, $y_1$ and $z_1$ —; $x_2$, $y_2$, and $z_2$ - - -), (d) synchronization errors $e_1$, $e_2$ and $e_3$.

# 4. Conclusions

In this paper, we have studied how to synchronize two modified Chua circuits with each other using active control. Also, we have attained some sufficient conditions for global synchronization between the drive and response continuous-time chaotic systems using rigorous control theory. The simulation results illustrate that two modified Chua circuits can be easily controlled using active control techniques. We note that the controller using the state-observer given in this paper is not the only possible form. We choose this form because matrix $\boldsymbol{K}$ could be simply expressed from Matlab. There are many observer design techniques and some of them may give better results in the synchronization of chaotic systems.

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