Vol 14 No 3, March 2005
1009-1963/2005/14(03)/0468-04

Chinese Physics

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Divergence structure of the statistical entropy of the Dirac field in a plane symmetry black hole geometry

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(Received 12 July 2004; revised manuscript received 5 September 2004)

The divergences at all levels for the statistical entropy of a plane symmetry black hole arising from the massless Dirac field are considered using the brick-wall model. It is shown that if we ignore the usual contribution from the vacuum surrounding the system, then the statistical entropy consists of two parts: one is the linearly divergent term which has the geometric character, the other consists of two logarithmically divergent terms which are not proportional to the surface area of the horizon. The entropy of the Dirac field on extremal plane symmetry spacetime background has higher divergence than usual.

Keywords: brick-wall model, statistical entropy, plane symmetry black hole, Dirac field
PACC: 0420, 9760L

1. Introduction

Employing the brick-wall model,[1] many researchers have studied the divergence structure of the statistical entropy of spin fields in various black hole backgrounds and pointed out that there exist logarithmically divergent terms that depends on the spin of the field considered and not only the degeneracy.[2−5] Therefore, the whole expression of the entropy of a spin field does not take the form of the scalar field. In this paper we try to calculate the statistical entropy of a plane symmetry black hole arising from the Dirac field using the brick-wall model. We begin in Section 2 by considering the Dirac field in this black hole background. In Section 3 we drive a basic formula via brick-wall method for the entropy. In Section 4 we calculate the black hole entropy. Conclusion is drawn in Section 5.

2. Dirac field equation

The metric outside the plane symmetry black hole can be written as:[6]

$$\mathrm{d}s^2 = B\mathrm{d}t^2 - B^{-1}\mathrm{d}r^2 - C(\mathrm{d}x^2 + \mathrm{d}y^2), \qquad (1)$$

where $1/2 < N < 2$, and

$$B = -\frac{4\pi M}{N\alpha^N} r^{1-N} + \frac{6\alpha^2}{N(2N-1)} r^N + \frac{2Q^2}{N\alpha^{2N}} r^{-N},$$
$$C = (\alpha r)^N. \qquad (2)$$

In curved spacetime, the massless Dirac spinor field equations read[7]

$$\begin{aligned}
&(D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 = 0,\\
&(\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = 0,\\
&(D + \varepsilon^* - \rho^*)G_2 + (\delta + \pi^* - \alpha^*)G_1 = 0,\\
&(\Delta + \mu^* - \gamma^*)G_1 + (\bar{\delta} + \beta^* - \tau^*)G_2 = 0, \qquad (3)
\end{aligned}$$

where $\Phi = (F_1, F_2, G_1, G_2)^{\mathrm{T}}$ is the four-component spinor, $D, \Delta, \delta, \bar{\delta}$ are ordinary differential functors, and $\alpha, \beta, \gamma, \cdots$ are Newman–Penrose spin coefficients, which are defined by[8]

$$\begin{aligned}
&D = l^\mu \partial_\mu, \quad \Delta = n^\mu \partial_\mu, \ \delta = m^\mu \partial_\mu,\\
&\kappa = l_{\mu;\nu} m^\mu l^\nu, \quad \pi = -n_{\mu;\nu} \bar{m}^\mu l^\nu, \quad \rho = l_{\mu;\nu} m^\mu \bar{m}^\nu,\\
&\lambda = -n_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu, \quad \sigma = l_{\mu;\nu} m^\mu m^\nu, \quad \mu = -n_{\mu;\nu} \bar{m}^\mu m^\nu,\\
&\tau = l_{\mu;\nu} m^\mu n^\nu, \quad \nu = -n_{\mu;\nu} \bar{m}^\mu n^\nu,\\
&\varepsilon = \frac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \bar{m}^\mu l^\nu),\\
&\alpha = \frac{1}{2}(l_{\mu;\nu} n^\mu \bar{m}^\nu - m_{\mu;\nu} \bar{m}^\mu \bar{m}^\nu),
\end{aligned}$$

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$$\beta = \frac{1}{2}(l_{\mu;\nu}n^\mu m^\nu - m_{\mu;\nu}\bar{m}^\mu m^\nu),$$
$$\gamma = \frac{1}{2}(l_{\mu;\nu}n^\mu n^\nu - m_{\mu;\nu}\bar{m}^\mu n^\nu), \quad (4)$$

where $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ is the null tetrad vector which satisfies

$$l^\mu l_\mu = n^\mu n_\mu = m^\mu m_\mu = \bar{m}^\mu \bar{m}_\mu = 0,$$
$$l^\mu m_\mu = l^\mu \bar{m}_\mu = n^\mu m_\mu = n^\mu \bar{m}_\mu = 0,$$
$$l^\mu n_\mu = -m^\mu \bar{m}_\mu = 1. \quad (5)$$

According to the conditions (5) and the metric (1), we can take the null tetrad vector as

$$l^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{B}}, -\sqrt{B}, 0, 0\right),$$
$$n^\mu = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{B}}, \sqrt{B}, 0, 0\right),$$
$$m^\mu = \frac{1}{\sqrt{2}}\left(0, 0, -\frac{1}{\sqrt{C}}, -\mathrm{i}\frac{1}{\sqrt{C}}\right),$$
$$\bar{m}^\mu = \frac{1}{\sqrt{2}}\left(0, 0, -\frac{1}{\sqrt{C}}, \mathrm{i}\frac{1}{\sqrt{C}}\right). \quad (6)$$

From the metric (1) and the null tetrad vector (6), we obtain the spin coefficients and ordinary differential functors using the Newman and Penrose formula (4):

$$\kappa = \pi = \varepsilon = \lambda = \sigma = \nu = \tau = \alpha = \beta = 0,$$
$$\rho = \mu = -\frac{\sqrt{B}C'}{2\sqrt{2}C}, \quad \gamma = \frac{-B'}{4\sqrt{2B}},$$
$$D = \frac{1}{\sqrt{2B}}\frac{\partial}{\partial t} - \frac{\sqrt{B}}{\sqrt{2}}\frac{\partial}{\partial r},$$
$$\Delta = \frac{1}{\sqrt{2B}}\frac{\partial}{\partial t} + \frac{\sqrt{B}}{\sqrt{2}}\frac{\partial}{\partial r},$$
$$\delta = -\frac{1}{\sqrt{2C}}\frac{\partial}{\partial x} - \frac{\mathrm{i}}{\sqrt{2C}}\frac{\partial}{\partial y},$$
$$\bar{\delta} = -\frac{1}{\sqrt{2C}}\frac{\partial}{\partial x} + \frac{\mathrm{i}}{\sqrt{2C}}\frac{\partial}{\partial y}. \quad (7)$$

In order to reduce the Dirac field equations (3), we introduce the following transformations on the Dirac field functions:

$$F_1 = \frac{1}{\sqrt{C}}R_{-1/2}(r)\mathrm{e}^{-\mathrm{i}Et}S_{-1/2}(x,y),$$
$$F_2 = R_{+1/2}(r)\mathrm{e}^{-\mathrm{i}Et}S_{+1/2}(x,y),$$
$$G_1 = R_{+1/2}(r)\mathrm{e}^{-\mathrm{i}Et}S_{-1/2}(x,y),$$
$$G_2 = \frac{1}{\sqrt{C}}R_{-1/2}(r)\mathrm{e}^{-\mathrm{i}Et}S_{+1/2}(x,y). \quad (8)$$

Substituting Eqs.(7) and (8) into the Dirac field equations (3), we obtain the following equations:

$$CB^2\frac{\mathrm{d}^2R_{-1/2}}{\mathrm{d}r^2} + \frac{(CB^2)'}{2}\frac{\mathrm{d}R_{-1/2}}{\mathrm{d}r} + \left[\frac{\mathrm{i}E}{2}(C'B - CB') + CE^2 + \frac{BB'C' + 2BCB''}{8} - B\lambda^2\right]R_{-1/2} = 0, \quad (9)$$

$$CB^2\frac{\mathrm{d}^2R_{+1/2}}{\mathrm{d}r^2} + \frac{(CB^2)'}{2}\frac{\mathrm{d}R_{+1/2}}{\mathrm{d}r} + \left[\frac{\mathrm{i}E}{2}(C'B - CB') + CE^2 + \frac{5BB'C' + 2BCB'' + 4B^2C'}{8} - B\lambda^2\right]R_{+1/2} = 0, \quad (10)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)S_{\pm1/2} + \lambda^2 S_{\pm1/2} = 0. \quad (11)$$

The solution of Eqs.(11) is a plane wave,[9] so the separation constant can be written as $\lambda^2 = m^2 + n^2$.

3. Basic formula

Using the WKB approximation with $R_{\pm1/2}(r) = \mathrm{e}^{\mathrm{i}S_\pm(r)}$, we find that Eqs.(9) and (10) yield the radial wave number $k(r,m,n,E) \equiv \partial_r S$:

$$k_{-1/2} = (CB^2)^{-1/2}\left[CE^2 + \frac{5B'C' + 2CB'' + 4BC''}{8}B - (m^2+n^2)B\right]^{1/2},$$
$$k_{+1/2} = (CB^2)^{-1/2}\left[CE^2 + \frac{B'C' + 2CB''}{8}B - (m^2+n^2)B\right]^{1/2}, \quad (12)$$

with the brick-wall boundary conditions: $R_{-1/2} = R_{+1/2} = 0$ at $r = r_\mathrm{h} + \varepsilon$ and $r = r_\mathrm{h} + L$, where $0 < \varepsilon \ll r_\mathrm{h}$, $L \gg r_\mathrm{h}$. Then the number of modes with energy smaller than E is

$$g(E) = \iint \mathrm{d}m\mathrm{d}n\frac{1}{\pi}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L}\left[2k_{+1/2}(r,m,n,E) + 2k_{-1/2}(r,m,n,E)\right]\mathrm{d}r$$
$$= \frac{4}{3}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L}\frac{C\mathrm{d}r}{B^2}\left\{\left[E^2 + \frac{(B'C' + 2CB'')B}{8C}\right]^{3/2} + \left[E^2 + \frac{(5B'C' + 2CB'' + 4BC'')B}{8C}\right]^{3/2}\right\}. \quad (13)$$

The free energy at inverse temperature β is given by

$$-\beta F = \sum_E \ln(1+\mathrm{e}^{-\beta E}). \tag{14}$$

Using Eq.(13) to determine the density of states, we obtain the free energy

$$\begin{aligned} F =& -\frac{1}{\beta}\int_0^\infty \mathrm{d}E \frac{\mathrm{d}g(E)}{\mathrm{d}E}\ln(1+\mathrm{e}^{-\beta E}) \\ =& -\int_0^\infty \frac{\mathrm{e}^{-\beta E}}{1+\mathrm{e}^{-\beta E}} g(E)\mathrm{d}E \\ =& -\frac{4}{3}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L} \frac{C}{B^2}\mathrm{d}r \int_0^\infty \frac{\mathrm{e}^{-\beta E}\mathrm{d}E}{1+\mathrm{e}^{-\beta E}} \\ & \times \left\{ \left[E^2 + \frac{(B'C'+2CB'')B}{8C} \right]^{3/2} \right. \\ & \left. + \left[E^2 + \frac{(5B'C'+2CB''+4BC'')B}{8C} \right]^{3/2} \right\} \\ =& -\frac{7}{3}\frac{\Gamma(4)\zeta(4)}{\beta^4}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L} \frac{C}{B^2}\mathrm{d}r \\ & -\frac{\zeta(2)}{\beta^2}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L} \frac{\mathrm{d}r}{B}\left[\frac{(BC)''}{2} - \frac{B'C'}{4} \right]. \end{aligned} \tag{15}$$

The entropy corresponding to Eq.(15) is

$$\begin{aligned} S =& \beta^2 \frac{\partial F}{\partial \beta} \\ =& \frac{28}{3}\frac{\Gamma(4)\zeta(4)}{\beta^3}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L} \frac{C}{B^2}\mathrm{d}r \\ & + \frac{\zeta(2)}{\beta}\int_{r_\mathrm{h}+\varepsilon}^{r_\mathrm{h}+L} \frac{\mathrm{d}r}{B}\left[(BC)'' - \frac{B'C'}{2} \right]. \end{aligned} \tag{16}$$

4. Statistical entropy

In the following we will use the formula (16) to study the statistical entropy of the Dirac field for nonextremal and extremal plane symmetry spacetime background.

4.1. Nonextremal spacetime background

In this case, B can be factorized into

$$B(r) = \frac{6\alpha^2}{N(2N-1)r^N}(r-r_\mathrm{h})\eta(r), \tag{17}$$

where r_h is the event horizon. The surface gravity of the horizon is

$$\kappa = \frac{B'(r_\mathrm{h})}{2} = \frac{3\alpha^2}{N(2N-1)}\frac{\eta(r_\mathrm{h})}{r_\mathrm{h}^N}. \tag{18}$$

Substituting Eq.(17) into Eq.(16) and ignoring the usual contribution from the vacuum surrounding the system as $'$t Hooft did, we can obtain the divergence of the entropy of the spin field with $\beta = \beta_H = 2\pi/\kappa$,

$$\begin{aligned} S_q =& \frac{7\Gamma(4)\zeta(4)A_\mathrm{h}}{12\pi^2\beta}\frac{1}{\varepsilon} \\ & + \frac{7\Gamma(4)\zeta(4)N^2(2N-1)^2}{27\beta^3\alpha^{4-N}}a_1 \\ & \times \ln\frac{L}{\varepsilon} + \frac{\zeta(2)}{\beta}b_1\ln\frac{L}{\varepsilon}, \end{aligned} \tag{19}$$

where $A_\mathrm{h} = C(r_\mathrm{h})$ is the area of the horizon corresponding to unit xoy-plane, and

$$\begin{aligned} a_1 =& \left(\frac{r^{3N}}{[\eta(r)]^2} \right)'\Big|_{r=r_\mathrm{h}} \\ b_1 =& \begin{cases} \dfrac{1}{\kappa}\left[\dfrac{3(7N-4)\alpha^{N+2}r^{2N-2}}{2(2N-1)} + \dfrac{Q^2}{2\alpha^N r_\mathrm{h}^2} - \dfrac{\pi M}{(N-1)r_\mathrm{h}} \right], & (N\neq 1) \\ \dfrac{1}{\kappa}\left[\dfrac{3(7N-4)\alpha^{N+2}r^{2N-2}}{2(2N-1)} + \dfrac{Q^2}{2\alpha^N r_\mathrm{h}^2} \right]. & (N=1) \end{cases} \end{aligned} \tag{20}$$

Eq.(19) shows that the entropy of the spin field has both linearly and logarithmically divergent terms.

In particular, when N=1, so long as $\pi^2M^2 > 3\alpha^2Q^2$, B can be written as

$$B(r) = \frac{6\alpha^2}{r}(r-r_\mathrm{h})(r-r_-), \tag{21}$$

$$\kappa = \frac{B'(r_\mathrm{h})}{2} = \frac{3\alpha^2(r_\mathrm{h}-r_-)}{r_\mathrm{h}}. \tag{22}$$

Where

$$\begin{aligned} r_\mathrm{h} &= (\pi M + \sqrt{\pi^2M^2 - 3\alpha^2Q^2})/3\alpha^2, \\ r_- &= (\pi M - \sqrt{\pi^2M^2 - 3\alpha^2Q^2})/3\alpha^2, \end{aligned} \tag{23}$$

r_- being the smaller positive root corresponding to the inner black hole horizon and $\mathrm{r_h}$ the larger positive root corresponding to the outer black hole horizon.

From Eq.(19), we can obtain the entropy

$$S_q = \frac{7\Gamma(4)\zeta(4)A_{\rm h}}{12\pi^2\beta}\frac{1}{\varepsilon} + \frac{7\Gamma(4)\zeta(4)}{27\beta^3\alpha^3}a_1 \times \ln\frac{L}{\varepsilon} + \frac{\zeta(2)}{\beta}b_1 \ln\frac{L}{\varepsilon}, \quad (24)$$

where

$$a_1 = \left(\frac{r^3}{(r-r_-)^2}\right)'\Big|_{r=r_{\rm h}} = 27\alpha^4\kappa^2 - 54\alpha^6\kappa^3,$$
$$b_1 = \frac{1}{\kappa}\left[\frac{9\alpha^3}{2} + \frac{Q^2}{2\alpha r_{\rm h}^2}\right]. \quad (25)$$

The leading term of the Eq.(24) is in agreement with the result of Ref.[10].

4.2. Extremal spacetime background

In this case, when N=1, and $\pi^2M^2 = 3\alpha^2Q^2$, B can be written as

$$B(r) = \frac{6\alpha^2}{r}(r - r_{\rm h})^2, \quad (26)$$

where $r_{\rm h} = \pi M/3\alpha^2$. This shows that the inner and outer horizons coincide, the black hole becomes extremal. Similar to the previous section, we can derive the entropy of the Dirac field on the extremal plane symmetry spacetime background

$$S_q = \frac{7\Gamma(4)\zeta(4)}{27\beta^3\alpha^3}\left[\frac{r_{\rm h}^3}{3\varepsilon^3} + \frac{3r_{\rm h}^2}{2\varepsilon^2} + \frac{3r_{\rm h}}{\varepsilon} + \ln\frac{L}{\varepsilon}\right] + \frac{\zeta(2)}{\beta}\left[\frac{2\alpha r_{\rm h}}{\varepsilon} + \frac{3\alpha^2-1}{2\alpha}\ln\frac{L}{\varepsilon}\right]. \quad (27)$$

In comparison with Eq.(24) we observe the additional cubic, quadratic divergences in Eq.(27).

5. Conclusion

We have investigated the entropy of the Dirac field on the nonextremal plane symmetry spacetime background using WKB approximation and then make use of the brick-wall model. The result (19) shows the entropy of the spin field has both linearly and logarithmically divergent terms, where the linear divergence can be reduced to the form proportional to the horizon area, but the logarithmical divergences depend on the black hole characteristics and therefore can not be regarded as a nonessential additive constant and neglected, so the whole entropy is not proportional to the scalar one.

We have also considered the Dirac field on the extremal plane symmetry spacetime background for N =1. The entropy is given by Eq.(27), which shows that the entropy of extremal black hole has higher divergence than usual. From Eq.(22) the Hawking temperature of the coinciding horizons vanishes. If this temperature is accepted, Eq.(27) for the entropy also vanishes, which satisfies Nernst's theorem.

References

[1] 't Hooft G 1985 *Nucl. Phys.* B **256** 727
[2] Li Zh H 2000 *Phys. Rev.* D **62** 024001
[3] Jing J L and Yan M L 2001 *Phys. Rev.* D **63** 084028
[4] Li Zh H 2002 *Mod. Phys. Lett.* A **17** 887
[5] Li G Q 2003 *Acta Phys. Sin.* **52** 1346 (in Chinese)
[6] Cai R G and Zhang Y Z 1996 *J. Phys. Rev.* D **54** 4891
[7] Teukolsky A S 1973 *Astrophys. J.* **185** 635
[8] Newman E and Penrose R 1962 *J. Math. Phys.* **3** 566
[9] Li Zh H and Zhao Zh 1997 *Acta Phys. Sin.* **46** 1273 (in Chinese)
[10] Zhao R and Zhang L Ch 2001 *Acta Phys. Sin.* **51** 21 (in Chinese)