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Bound states of Klein–Gordon equation for double ring-shaped oscillator scalar and vector potentials*

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In spherical polar coordinates, double ring-shaped oscillator potentials have supersymmetry and shape invariance for θ and r coordinates. Exact bound state solutions of Klein–Gordon equation with equal double ring-shaped oscillator scalar and vector potentials are obtained. The normalized angular wavefunction expressed in terms of Jacobi polynomials and the normalized radial wavefunction expressed in terms of the Laguerre polynomials are presented. Energy spectrum equations are obtained.

Keywords: double ring-shaped oscillator potentials, supersymmetry and shape invariance, Klein–Gordon equation, scalar and vector potentials, bound states

PACC: 0365G, 1110Q, 1240Q

1. Introduction

When a particle moves in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics.[1] The Klein–Gordon equation can describe relativistic particles with zero or integer spin. In recent years, some authors have assumed that the scalar potential is equal to vector potential, and obtained the s-wave bound states or scattering solution of particles of the Klein–Gordon and Dirac equation in some typical potential fields. These typical potential fields include Hulthén potential,[2,3,4] Morse and Wood–Saxon potential,[5,6] harmonic oscillator potential,[7] Pöschl–Teller potential,[8,9,10] $\tan^2(\pi\eta r)$-type potential[11] and N-dimensional hydrogen potentials.[12] All the studies mentioned above are devoted to the problems concerning the centrally symmetric potential fields. Supersymmetry was first studied in the simplest case of supersymmetric quantum mechanics by Witten,[13] and Cooper and Freeman.[14] It has been realized that the supersymmetry gives insight into the factorization method of Infeld and Hull.[15] In 1983, Gendenstein introduced the concept of shape-invariable potential.[16] It is readily shown that for any shape-invariable potential the energy eigen spectrum and energy eigenfunction can be obtained algebraically. With the ideas of supersymmetry and shape invariance, a number of potential problems can be solved exactly. Although most of these potentials are either one-dimensional or centrally symmetrical, some non-centrally-symmetrical separable potentials are involved. Those potentials have supersymmetry and shape invariance and can be exactly solved in terms of the supersymmetric quantum mechanics and shape invariance.

The double ring-shaped oscillator (DRSO) potential, a three-dimensional harmonic oscillator potential surrounded by a double ring-shaped inverse square potential, can describe the properties of nucleus. In spherical coordinates, the DRSO potential, one of the non-centrally-supersymmetric potentials, is given by the following expression:

$$V(r,\theta)=\frac{1}{2}kr^2+\frac{b}{r^2\sin^2\theta}+\frac{a}{r^2\cos^2\theta},\quad (a>0,b>0), \tag{1}$$

where k is the elastic coefficient, a and b are positive real parameters. When $a=0$ and $b=0$, the DRSO

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potential reduces to a harmonic oscillator (HO) potential, which is one of the most important models in classical and quantum physics and is widely used in many branches. When $a = 0$, the DRSO potential reduces to a ring-shaped oscillator (RSO) potential, and the bound states of the Schrödinger equation for the RSO potential have been studied by using the supersymmetry and shape invariance[17] and series expansion method.[18] The bound states of the Schrödinger equation for the DRSO have been investigated by using the supersymmetry and shape invariance method.[19] The exact solution of the Klein–Gordon equation with equal RSO scalar and vector potentials has been obtained.[20,21] In this paper, we show that the DRSO potentials have supersymmetry and shape invariance for θ and r coordinates, and acquire the bound states of the Klein–Gordon equation with equal DRSO scalar and vector potentials. The normalized wavefunction and the exact energy spectrum equations are also presented.

This paper is organized as follows. In Section 2, the separation of variables is carried out for the Klein–Gordon equation with equal DRSO scalar and vector potentials in spherical coordinates. In Section 3, we obtain the normalized angular wavefunction expressed in terms of the Jacobi polynomials. The normalized radial wavefunction and exact energy spectrum equation of bound states are presented in Section 4.

2. Separation of variables for the Klein–Gordon equation

The Klein–Gordon equation with scalar and vector potentials is given by ($\hbar = c = 1$)[22]

$$\begin{aligned}&\left\{P^2+[\mu + S(r,\theta,\varphi)/2]^2\right\}\psi(r,\theta,\varphi)\\ =&[E - V(r,\theta,\varphi)/2]^2\psi(r,\theta,\varphi),\end{aligned} \tag{2}$$

where P is the momentum operator, μ and E are the rest mass energy of the particle, including the rest mass and energy, respectively. The Klein–Gordon equation with equal scalar and vector potentials is given by

$$\begin{aligned}&[P^2 + (E+\mu)V(r,\theta,\varphi)]\psi(r,\theta,\varphi)\\ =&(E^2-\mu^2)\psi(r,\theta,\varphi).\end{aligned} \tag{3}$$

In spherical polar coordinates, the most general potential for which the Klein–Gordon equation is variable-separable, is given by[23]

$$V(r,\theta,\varphi) = V(r) + \frac{V(\theta)}{r^2} + \frac{V(\varphi)}{r^2\sin^2\theta}, \tag{4}$$

where $V(r)$, $V(\theta)$ and $V(\varphi)$ are arbitrary functions of a certain argument. It is convenient to write the wavefunction as

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r}\frac{H(\theta)}{(\sin\theta)^{1/2}}K(\varphi). \tag{5}$$

Substituting Eq.(5) into Eq.(3), and using the standard procedure of separation of variables, one obtains the following equations for the function $K(\varphi)$, $H(\theta)$ and $R(r)$ respectively:

$$-\frac{\mathrm{d}^2K(\varphi)}{\mathrm{d}\varphi^2} + (E+\mu)V(\varphi)K(\varphi) = m^2K(\varphi), \tag{6}$$

$$\begin{aligned}&-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + \left[(E+\mu)V(\theta) + \frac{m^2-1/4}{\sin^2\theta}\right]H(\theta)\\ =&l^2H(\theta),\end{aligned} \tag{7}$$

$$\begin{aligned}&-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \left[(E+\mu)V(r) + \frac{l^2-1/4}{r^2}\right]R(r)\\ =&(E^2-\mu^2)R(r),\end{aligned} \tag{8}$$

where m^2 and l^2 are separation constants. For the DRSO potential, we have

$$V(r) = \frac{1}{2}kr^2, \quad V(\theta) = \frac{b}{\sin^2\theta} + \frac{a}{\cos^2\theta}, \quad V(\varphi) = 0, \tag{9}$$

then, the following three equations are obtained:

$$-\frac{\mathrm{d}^2K(\varphi)}{\mathrm{d}\varphi^2} = m^2K(\varphi), \tag{10}$$

$$\begin{aligned}&-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2} + \left[\frac{(E+\mu)b + m^2 - 1/4}{\sin^2\theta}\right.\\ &\left.+\frac{(E+\mu)a}{\cos^2\theta}\right]H(\theta) = l^2H(\theta),\end{aligned} \tag{11}$$

$$\begin{aligned}&-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \left[\frac{1}{2}kr^2(E+\mu) + \frac{l^2-1/4}{r^2}\right]R(r)\\ =&(E^2-\mu^2)R(r).\end{aligned} \tag{12}$$

The periodic boundary condition for Eq.(10) is given by $K(\varphi + 2\pi) = K(\varphi)$, and it is well-known that the solution of Eq.(10) is given by

$$K(\varphi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}m\varphi}, \quad |m| = 0,1,2\cdots. \tag{13}$$

In the following two sections we will use the supersymmetry and shape invariance to acquire the energy spectrum and wavefunction of the DRSO potential for θ and r coordinates.

3. Supersymmetry and shape invariance for θ coordinate

Letting

$$B=\sqrt{(E+\mu)b+m^2}+1/2,$$
$$A=\sqrt{(E+\mu)a+1/4}+1/2,\quad \lambda=l^2, \tag{14}$$

Eq.(11) becomes

$$-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2}+\left[\frac{B(B-1)}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta}\right]H(\theta)=\lambda H(\theta), \tag{15}$$

whose ground state function and eigenvalue are given respectively by

$$H_0(\theta)=(\sin\theta)^B(\cos\theta)^A,\quad \lambda_0=l_0^2=(A+B)^2. \tag{16}$$

The corresponding superpotential is given by

$$W(\theta,A,B)=A\tan\theta-B\cot\theta. \tag{17}$$

Then, we can obtain the partner potentials,

$$\begin{aligned}V_-(\theta,A,B)=&W^2-W'\\=&-(A+B)^2+A(A-1)\sec^2\theta\\&+B(B-1)\csc^2\theta,\end{aligned} \tag{18}$$

$$\begin{aligned}V_+(\theta,A,B)=&W^2+W'\\=&-(A+B)^2+A(A+1)\sec^2\theta\\&+B(B+1)\csc^2\theta.\end{aligned} \tag{19}$$

It is deduced that

$$V_+(\theta,A,B)=V_-(\theta,A+1,B+1)+R(A+1,B+1), \tag{20}$$

where

$$R(A+1,B+1)=(A+B+2)^2-(A+B)^2. \tag{21}$$

Hence, the shape invariance condition is satisfied for θ coordinate, and the complete eigenvalue spectrum is given by

$$\begin{aligned}\lambda_s^-=&\sum_{k=1}^{s}R(A+k,B+k)\\=&(A+B+2s)^2-(A+B)^2.\end{aligned} \tag{22}$$

Thus, one can have the value of λ in Eq.(15) as follows,

$$\lambda=l^2=(A+B+2s)^2,$$
$$l=A+B+2s,\quad s=1,2,3\cdots. \tag{23}$$

In principle, we can use supersymmetric quantum mechanics and shape invariance to find the eigenfunction of Eq.(15), here an alternative method is used. Substituting Eq.(23) into Eq.(15) yields

$$\begin{aligned}&-\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2}+\left[\frac{B(B-1)}{\sin^2\theta}+\frac{A(A-1)}{\cos^2\theta}\right]H(\theta)\\=&(A+B+2s)^2H(\theta).\end{aligned} \tag{24}$$

Letting

$$H(\theta)=(\sin\theta)^B(\cos\theta)^AY(\theta) \tag{25}$$

and substituting Eq.(25) into Eq.(24) gives

$$\begin{aligned}&\frac{\mathrm{d}^2Y(\theta)}{\mathrm{d}\theta^2}+2(B\cot\theta-A\tan\theta)\frac{\mathrm{dY}(\theta)}{\mathrm{d}\theta}\\&+4s(A+B+s)Y(\theta)=0.\end{aligned} \tag{26}$$

When a new variable $x=\sin^2\theta$ is introduced, a Jacobi differential equation is obtained:[24]

$$\begin{aligned}&x(1-x)\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}+[(B+1/2)-(A+B+1)x]\\&\times\frac{\mathrm{d}Y}{\mathrm{d}x}+s(A+B+s)Y=0.\end{aligned} \tag{27}$$

The solution of Eq.(27) are Jacobi polynomials,

$$Y(x)=F(-s,p+s,\gamma,x),\quad p=A+B,\gamma=B+1/2, \tag{28}$$

where $F(\alpha,\beta,\gamma,x)$ is the hypergeometric function. The angular wavefunction can be written as

$$H(\theta)=N_s(\sin\theta)^B(\cos\theta)^AF(-s,p+s,\gamma,x), \tag{29}$$

where N_s is the normalization constant of angular wavefunction. According to the normalized condition $\int_0^{\pi}H^2(\theta)\mathrm{d}\theta=2\int_0^{\pi/2}H^2(\theta)\mathrm{d}\theta=1$ and using the relation of the orthogonality of the Jacobi polynomials[24]

$$\begin{aligned}&\int_0^1 z^{\gamma-1}(1-z)^{p-\gamma}F^2(-n,p+n,\gamma,z)\mathrm{d}z\\=&\frac{n!}{p+2n}\times\frac{\Gamma^2(\gamma)\Gamma(p+n-\gamma+1)}{\Gamma(p+n)\Gamma(\gamma+n)},\end{aligned} \tag{30}$$

we have

$$N_s=\left\{\frac{p+2s}{s!}\times\frac{\Gamma(p+s)\Gamma(\gamma+s)}{\Gamma^2(\gamma)\Gamma(p+s-\gamma+1)}\right\}^{1/2}. \tag{31}$$

So the normalized angular wavefunction is

$$\begin{aligned}H(\theta)=&\left\{\frac{A+B+2s}{s!}\right.\\&\left.\times\frac{\Gamma(A+B+s)\Gamma(B+s+1/2)}{\Gamma^2(B+1/2)\Gamma(A+s+1/2)}\right\}^{1/2}\\&\times(\sin\theta)^B(\cos\theta)^A\\&\times F(-s,A+B+s,B+1/2,\sin^2\theta).\end{aligned} \tag{32}$$

4. Supersymmetry and shape invariance for r coordinate and energy spectrum

Substituting the notation

$$\alpha=\sqrt{(E+\mu)k/2},\quad l'=l-1/2,\quad \lambda=E^2-\mu^2, \tag{33}$$

into Eq.(12), the radial part of the Klein–Gordon equation can be written as

$$-\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\left[\alpha^2r^2+\frac{l'(l'+1)}{r^2}\right]R(r)=\lambda R(r), \tag{34}$$

whose ground state function and ground eigenvalue, respectively are

$$R_0(r)=r^{l'+1}\exp\left(\frac{-\alpha r^2}{2}\right),\quad \lambda_0=2\alpha(l'+3/2). \tag{35}$$

The corresponding superpotential is

$$W(r)=\alpha r-\frac{l'+1}{r}. \tag{36}$$

Then, the partner potential is obtained as follows

$$\begin{aligned}V_-(r,l')=&W^2-W'\\=&\alpha^2r^2+\frac{l'(l'+1)}{r^2}-2\alpha(l'+3/2),\end{aligned} \tag{37}$$

$$\begin{aligned}V_+(r,l')=&W^2+W'\\=&\alpha^2r^2+\frac{(l'+1)(l'+2)}{r^2}-2\alpha(l'+1/2).\end{aligned} \tag{38}$$

It is deduced that

$$V_+(r,l')=V_-(r,l'+1)+R(l'+1),R(l'+1)=4\alpha. \tag{39}$$

Hence, the shape invariance condition is satisfied for r coordinate, and the complete eigenvalue spectrum is given by

$$\lambda_n^-=\sum_{k=1}^{n}R(l'+k)=4n\alpha,n=1,2,3,\cdots, \tag{40}$$

Thus, we have the value of λ in Eq.(34) as follows:

$$\lambda_n=2\alpha(2n+l'+3/2). \tag{41}$$

From Eqs.(33) and (41), the energy spectrum equation is obtained as follows:

$$E^2-\mu^2=2\alpha(2n+l'+3/2) \tag{42}$$

and

$$(E-\mu)\sqrt{E+\mu}=\sqrt{2k}(2n+A+B+2s+1). \tag{43}$$

In principle, we can use supersymmetric quantum mechanics and shape invariance to find the eigenfunction of Eq.(34), here an alternative method is used. Letting

$$R(r)=r^{l'+1}\exp\left(\frac{-\alpha r^2}{2}\right)L(r) \tag{44}$$

and substituting Eqs.(41) and (44) into Eq.(34) gives

$$\frac{\mathrm{d}^2L(r)}{\mathrm{d}r^2}+2\left(-\alpha r+\frac{l'+1}{r}\right)\frac{\mathrm{d}L(r)}{\mathrm{d}r}+4n\alpha L(r)=0. \tag{45}$$

Introducing a new variable $y=\alpha r^2$, Eq.(45) can be rewritten as

$$y\frac{\mathrm{d}^2L(y)}{\mathrm{d}y^2}+\left(l'+\frac{3}{2}-y\right)\frac{\mathrm{d}L(y)}{\mathrm{d}y}+nL(y)=0, \tag{46}$$

which is just the well-known differential equation satisfied by Laguerre polynomials, and the analytic solutions of Eq.(46) can be expressed as the Laguerre polynomials[24]

$$L(y)=L_n^\nu(y),\quad \nu=l'+1/2. \tag{47}$$

The radial wavefunction can be obtained as follows:

$$R_{nl'}(r)=N_{nl'}r^{l'+1}\exp\left(\frac{-\alpha r^2}{2}\right)L_n^\nu(\alpha r^2), \tag{48}$$

where $N_{nl'}$ is normalization constant of radial wavefunction. Using the normalization condition $\int_0^\infty R^2(r)\mathrm{d}r=1$ and the relation of the orthogonality of Laguerre polynomials[24]

$$\int_0^\infty z^\nu \mathrm{e}^{-z}L_n^\nu(z)L_{n'}^\nu(z)\mathrm{d}z=\frac{\Gamma(n+\nu+1)}{n!}\delta_{nn'}, \tag{49}$$

one can derive the normalization constant

$$N_{nl'}=\sqrt{\frac{2\alpha^{l'+3/2}n!}{\Gamma(n+l'+3/2)}}. \tag{50}$$

So the normalized radial wavefunction of bound states for Klein–Gordon equation with equal scalar and vector potentials is given by

$$\begin{aligned}R_{nl'}(r)=&\sqrt{\frac{2\alpha^{l'+3/2}n!}{\Gamma(n+l'+3/2)}}r^{l'+1}\\&\times\exp\left(\frac{-\alpha r^2}{2}\right)L_n^{l'+1/2}(\alpha r^2).\end{aligned} \tag{51}$$

5. Conclusions

The above results show that some non-centrally-symmetrical potential problems of relativistic quantum mechanics can be also solved exactly. We have

obtained exact bound state solutions of Klein–Gordon equation with equal scalar and vector potentials, which are equal to half of the DRSO potentials, by using the method of supersymmetry and shape invariance. The normalized wavefunctions expressed respectively in terms of the product of Jacobi polynomials and Laguerre polynomials are presented. Wavefunction and energy spectrum equations of system are related to three quantum numbers (m, s and n) and the parameters of the DRSO potential (k, a and b).

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