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Mapping of the position-dependent mass Schrödinger equation under point canonical transformation

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In this paper, the three-dimensional radial position-dependent mass Schrödinger equation is exactly solved through mapping this wave equation into the constant mass Schrödinger equation with Coulomb potential by means of point canonical transformation. The wavefunctions here can be given in terms of confluent hypergeometric functions.

Keywords: position-dependent mass, Coulomb potential, point canonical transformation
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1. Introduction

In recent years, there has been growing interest in the investigations of the position-dependent mass Schrödinger equation,[1−17] because solutions to this equation are comprehensive applications in different fields of material science and condensed matter physics such as semiconductors,[18] quantum well and quantum dots,[19] ^{3}He clusters,[20] quantum liquids,[21] graded alloys and semiconductor heterostructures,[22] etc.

The point canonical transformation (PCT) approach is one of the most important methods for solving the Schrödinger equation with constant mass in non-relativistic quantum mechanics.[23] Recently, this method was extended to solve the position-dependent mass Schrödinger equation and the exact solutions of the well-known solvable one-dimensional equations were successfully obtained[4,8] and the three-dimensional harmonic oscillator was also discussed.[4]

In this paper, we will employ the PCT approach to solve the three-dimensional position-dependent mass Schrödinger equation with Coulomb potential. By mapping this equation into the constant mass Schrödinger equation, its exact bound state solutions are obtained.

2. Action of the PCT map on the three-dimensional Schrödinger equation

The three-dimensional position-dependent effective mass Schrödinger equation with physical potentials is given by[4,17]

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{m'}{m}\left(\frac{1}{r}-\frac{\mathrm{d}}{\mathrm{d}r}\right)-\frac{l(l+1)}{r^2}+2m[E-V(r)]\right\}R(r)=0, \tag{1}$$

where E is the energy, l is the angular momentum, $V(r)$ is the real potential function and $m'=\mathrm{d}m(r)/\mathrm{d}r$. On the other hand, the three-dimensional radial Schrödinger equation with constant mass, potential function $U(\rho)$, angular momentum Λ and energy ε is given by

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}-\frac{\Lambda(\Lambda+1)}{\rho^2}+2[\varepsilon-U(\rho)]\right\}\psi(\rho)=0. \tag{2}$$

Performing the following transformations on Eq.(2)

$$\rho=q(r),\quad \psi(\rho)=g(r)R(r), \tag{3}$$

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we obtain

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\left(2\frac{g'}{g}-\frac{q''}{q}\right)\frac{\mathrm{d}}{\mathrm{d}r}+\left(\frac{g''}{g}-\frac{q''}{q'}\frac{g'}{g}\right)-\Lambda(\Lambda+1)\left(\frac{q'}{q}\right)^2+2(q')^2[\varepsilon-U(q(r))]\right\}R(r)=0. \quad (4)$$

Comparing Eq.(4) with Eq.(1) we obtain the following conditions:

$$g(r)=\sqrt{q'/m}, \quad (5)$$

$$V(r)-E+\frac{l(l+1)}{2mr^2}=\frac{(q')^2}{m}[U(q(r))-\varepsilon]+\frac{m'}{2m^2r}+\frac{1}{2m}\Lambda(\Lambda+1)\left(\frac{q'}{q}\right)^2+\frac{1}{4m}[F(m)-F(q')], \quad (6)$$

where

$$F(z)=\frac{z''}{z}-\frac{3}{2}\left(\frac{z'}{z}\right)^2. \quad (7)$$

It will be shown in next section that for a given mass $m(r)$ we choose a transformation function $q(r)$ to determine $g(r)$ in terms of Eq.(5) and then deduce the energy spectrum E and function $V(r)$ by means of Eq.(6).

3. Application to the Coulomb potential

The Coulomb potential is given by[24]

$$U(\rho)=-\frac{Z}{\rho}. \quad (8)$$

By substituting Eq.(8) into Eq.(2), the bound state energy spectrum and wavefunction are respectively given by[24]

$$\varepsilon_n=-\frac{Z^2}{2(\Lambda+n+1)^2}, \quad (9)$$

$$\psi_n(\rho)=B_n(\beta\rho)^{\Lambda+1}\exp[-\beta\rho]F(-n,2\Lambda+1,2\beta\rho), \quad (10)$$

where $F(\ \ )$ denotes hypergeometric functions, B_n is the normalized constant and

$$\beta=\frac{z}{\Lambda+n+1}. \quad (11)$$

In this paper, we consider the position-dependent mass $m(r)=\alpha r^\lambda$ and use the PCT function $q(r)=r^v$, where α, λ and v are nonzero real parameters. For simplicity, only the case $v=1+\lambda/2$ with $\lambda\neq 2$, i.e. $(q')^2/m=$constant, is considered. Substituting them into Eqs.(5) and (6), we obtain

$$V(r)=-\frac{P}{r^{1+\lambda/2}}, \quad (12)$$

$$E_n=-\frac{2\alpha P^2}{(2+\lambda)^2}\frac{1}{(\Lambda(l)+n+1)^2}, \quad (13)$$

$$R_n(r)=b_n r^{(\Lambda(l)+1)\left(1+\frac{\lambda}{2}\right)+\frac{\lambda}{4}}\exp[-\gamma r^{1+\lambda/2}]\times F(-n,2\Lambda(l)+1,2\gamma r^{1+\lambda/2}), \quad (14)$$

where P is a real potential parameter,

$$\gamma=\frac{4\alpha P}{(2+\lambda)^2}\frac{1}{\Lambda(l)+n+1}, \quad (15)$$

and $\Lambda(l)$ can be calculated from the following equation,

$$16l(l+1)=4(2+\lambda)^2\Lambda(\Lambda+1)-3\lambda^2+12\lambda. \quad (16)$$

In order to check the results given by Eqs.(12)–(16), we discuss some cases of real parameter λ:

Case 1 For $\lambda=0$, i.e. m is independent of position, the potential function, energy spectrum and wavefunction can be expressed as

$$V(r)=-\frac{P}{r},\quad E_n=-\frac{\alpha P^2}{2(l+n+1)^2},\quad R_n(r)=b_n r^{l+1}\exp[-\gamma_1 r]F(-n,2l+1,2\gamma_1 r), \quad (17)$$

where $\gamma_1=\alpha P/(l+n+1)$.

Case 2 For $\lambda=1$, the potential function, energy spectrum and wavefunction can be expressed as

$$V(r)=-\frac{P}{r^{3/2}},\quad E_n=-\frac{2\alpha P^2}{9(\Lambda_2(l)+n+1)^2},\quad R_n(r)=b_n r^{\frac{3}{2}(\Lambda_2(l)+1)+\frac{1}{4}}\exp[-\gamma_2 r^{3/2}]\times F(-n,2\Lambda_2(l)+1,2\gamma_2 r^{3/2}), \quad (18)$$

where $\gamma_2=\dfrac{4\alpha P}{9(\Lambda_2(l)+n+1)}$ and $\Lambda_2(l)$ can be calculated from following equation,

$$16l(l+1)=36\Lambda_2(\Lambda_2+1)+9.$$

Case 3 For $\lambda=2$, the potential function, energy spectrum and wavefunction can be expressed as

$$V(r)=-\frac{P}{r^2},\quad E_n=-\frac{\alpha P^2}{8(\Lambda_3(l)+n+1)^2},\quad R_n(r)=b_n r^{2(\Lambda_3(l)+1)+\frac{1}{2}}\exp[-\gamma_3 r^2]\times F(-n,2\Lambda_3(l)+1,2\gamma_3 r^2), \quad (19)$$

where $\gamma_3 = \dfrac{\alpha P}{4(\Lambda_3(l)+n+1)}$ and $\Lambda_3(l)$ can be calculated from following equation,

$$4l(l+1) = 16\Lambda_3(\Lambda_3+1)+3.$$

4. Conclusions

In this paper, we have investigated the three-dimensional position-dependent mass Schrödinger equation. It has been shown that by means of a proper transformation, we can map this wave equation into the constant mass Schrödinger equation with the Coulomb potential, and then, the energy spectrum and wavefunctions can be obtained exactly.

References

[1] Dekar L, Chetouani L and Hammann T F 1998 *J. Math. Phys.* **39** 2551
Dekar L, Chetouani L and Hammann T F 1998 *Phys. Rev.* A **59** 107
[2] Gönül B *et al* 2002 *Mod. Phys. Lett.* A **17** 2057
[3] Gönül B *et al* 2002 *Mod. Phys. Lett.* A **17** 2453
[4] Alhaidari A D 2002 *Phys. Rev.* A **66** 042116
[5] Alhaidari A D 2003 *Int. J. Theor. Phys.* **42** 12
[6] Alhaidari A D 2004 *Phys. Lett.* A **322** 72
[7] Yu J, Dong S H and Sun G H 2004 *Phys. Lett.* A **322** 290
[8] Yu J and Dong S H 2004 *Phys. Lett.* A **325** 194
[9] Plastino A R 1999 *Phys. Rev.* A **60** 4318
[10] Samani K and Loran F 2003 quant-ph/0302191
[11] Milanović V and Ikonić Z 1999 *J. Phys.* A **32** 7001
[12] de Souza Dutra A and Almeida C A S 2000 *Phys. Lett.* A **275** 25
[13] Roy B and Roy P 2002 *J. Phys.* A **35** 3961
[14] Koc R, Koca M and Körcük E 2002 *J. Phys.* A **35** L527
[15] Koc R and Koca M 2003 *J. Phys.* A **36** 8105
[16] Quesne C and Tkachuk V M 2004 *J. Phys.* A **37** 4267
[17] Chen G 2004 *Phys. Lett.* A **329** 22
[18] Bastard G 1988 *Wave Mechanics Applied to Semiconductor Heterostructure* (Les Ulis: Les Editions de Physique)
[19] Harrison P 2000 *Quantum Wells, Wires and Dots* (New York: Wiley)
Serra L and Lipparin E 1997 *Europhys. Lett.* **40** 667
[20] Barranco M *et al* 1997 *Phys. Rev.* B **56** 8997
[21] Arias F *et al* 1997 *Phys. Rev.* B **50** 4248
[22] Weisbuch C and Vinter B 1993 *Quantum Semiconductor Heterostructure* (New York: Academic) and references therein
[23] De R, Dutt R and Sukhatme U 1992 *J. Phys.* A **25** L843
[24] Zeng J Y 1997 *Quantum Mechanics* Vol II, 2nd (Beijing: Science Press) (in Chinese)