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Exact travelling wave solutions to some fifth order dispersive nonlinear wave equations

Hu Jian-Lan(胡建兰)†

College of Applied Sciences, Beijing University of Technology, Beijing 100022, *China*

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Exact travelling wave solutions to some nonlinear equations of fifth order derivatives are derived by using some accurate ansatz methods.

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1. Introduction

The dynamical behaviour of a complicated system has been considered as one of the frontier subjects in the study of nonlinear dynamics. It is concerned with not only the behaviour of time evolution, but also the structure of space. Thus, the research is difficult. Cross and Hohenberg[1] presented a good review of the research on this problem and pointed out that there are four ways to study it, i.e. 1) numerical simulation, 2) qualitative analysis, 3) perturbation method and 4) finding exact solutions. During the last several decades, great progress has been made in the construction of exact solutions of integrable and nonintegrable nonlinear systems.

At present, there are several methods to find exact or approximate solutions, such as 1) the inverse scattering method,[2,3] 2) the Backlünd transformations,[4,5] 3) the Painlevé analysis,[6] 4) the Lie group method,[7] 5) the direct algebraic method and rank analysis method.[8−14] It is worth mentioning that the so-called rank analysis method is a powerful tool to construct exact solutions to nonintegrable dynamical systems. Meanwhile, it is simple to realize and practical as well.

In this paper, by proposing some new ansatz, we find exact solutions to some fifth order dispersive nonlinear wave equations:

(A) the second equation in the modified Korteweg–de Vries (KdV) hierarchy (SEMKdV)[15]

$$u_t + \partial_x(u_{4x} + 10u(u_x)^2 + 10u^2u_{xx} + 6u^5) = 0, \quad \text{(A)}$$

(B) water wave equation (WWE)[16]

$$u_t + u_x + c_1uu_x + c_2u_{3x} + c_3u_xu_{xx} + c_4uu_{3x} + c_5u_{5x} + c_6u^2u_x = 0, \quad \text{(B)}$$

where c_j are real constants.

Here the travelling wave solutions to the above listed equations are considered to be in the form $u(x,t) = u(\xi)$ with $\xi = x - \lambda t$, where λ stands for the wave speed, and they will be detailed in the next section.

2. Approach and exact solutions

2.1. Exact solutions to SEMKdV

Inserting $u(\xi)$ with $\xi = x - \lambda t$ into Eq.(A) yields the following reduced ordinary differential equation

$$-\lambda u + u^{(4)} + 10uu'^2 + 10u^2u'' + 6u^5 = 0\,. \quad (1)$$

Assume that

$$u'' = au^3 + bu^2 + cu, \quad (2)$$

where a, b and c are real constants to be determined. Then,

$$\begin{aligned} u^{(4)} =& 6a^2u^5 + 10abu^4 + 10\left(\frac{b^2}{3} + ac\right)u^3 \\ &+ 5bcu^2 + c^2u, \\ u'^2 =& \frac{a}{2}u^4 + \frac{2b}{3}u^3 + cu^2. \end{aligned} \quad (3)$$

†E-mail: jianlan0712@sina.com

Inserting Eqs.(2) and (3) into Eq.(1), we have

$$2a^2+5a+2=0,\ b\left(a+\frac{5}{3}\right)=0,\ \lambda-c^2=0,$$

which yield two sets of solutions:

$$a=-2,\quad b=0,\quad \lambda=c^2;$$
$$a=-\frac{1}{2},\quad b=0,\quad \lambda=c^2.$$

Thus, the expected solutions are

$$u=\pm\sqrt{c}\ \mathrm{sech}[\sqrt{c}(\xi-\xi_0)];\\ \pm 2\sqrt{c}\ \mathrm{sech}[\sqrt{c}(\xi-\xi_0)] \tag{4}$$

with c being an arbitrary positive constant and $\lambda=c^2$.

2.2. Exact solutions to WWE

Substituting $u(x,t)=u(\xi)$ with $\xi=x-\lambda t$ into Eq.(B) and integrating the resultant we obtain

$$(1-\lambda)u+\frac{c_1}{2}u^2+c_2u''+\frac{c_3-c_4}{2}u'^2\\ +c_4uu''+c_5u^{(4)}+\frac{c_6}{3}u^3=0, \tag{5}$$

with the integration constant neglected. Consider the following two cases.

Case 1 $c_6\neq 0$ in Eq.(5). In this case, we can assume u to be in the following form

$$u''=bu^2+cu, \tag{6}$$

which is a candidate for the travelling wave solution to Eq.(5). Inserting Eq.(6) into Eq.(5) gives the following algebraic equations:

$$p_1+p_4c+p_6c^2=0,$$
$$p_2+p_3c+p_4b+p_5c+5p_6bc=0,$$
$$p_7+\frac{10}{3}p_6b^2+p_5b+\frac{2}{3}bp_3=0,$$

where $p_7=\frac{c_6}{3}$, $p_6=c_5$, $p_5=c_4$, $p_4=c_2$, $p_3=(c_3-c_4)/2$, $p_2=c_1/2$ and $p_1=1-\lambda$. Solving the above algebraic equations we have the following sets of solutions

$$b=-\frac{p_3+p_5}{5p_6},\\ p_2=\frac{p_4(p_3+p_5)}{5p_6},\\ p_7=\frac{p_5(p_3+p_5)}{15p_6},\\ \lambda=1+p_4c+p_6c^2, p_6\neq 0. \tag{7}$$

$$b=-\frac{1}{20p_6}(2p_3+3p_5+\alpha),\\ c=\frac{1}{5}\left(-\frac{p_4}{p_6}+\frac{p_3^2p_4}{p_6^2\beta}+\frac{3p_3p_4p_5}{2p_6^2\beta}+\frac{p_4p_5^2}{2p_6^2\beta}\right.\\ \left.-\frac{5p_2p_3}{p_6\beta}-\frac{5p_2p_5}{2p_6\beta}+\frac{(p_3p_4+p_4p_5+5p_2p_6)\alpha}{2p_6^2\beta}\right),\\ \lambda=1+p_4c+p_6c^2, p_6\neq 0,\\ -p_3p_5-p_5^2+15p_6p_7\neq 0. \tag{8}$$

$$b=\frac{1}{20p_6}(-2p_3-3p_5+\alpha),\\ c=\frac{1}{5}\left(-\frac{p_4}{p_6}+\frac{p_3^2p_4}{p_6^2\beta}+\frac{3p_3p_4p_5}{2p_6^2\beta}+\frac{p_4p_5^2}{2p_6^2\beta}\right.\\ \left.-\frac{5p_2p_3}{p_6\beta}-\frac{5p_2p_5}{2p_6\beta}-\frac{(p_3p_4+p_4p_5-5p_2p_6)\alpha}{2p_6^2\beta}\right),\\ \lambda=1+p_4c+p_6c^2,\\ p_6\neq 0,\\ -(p_3p_5)-p_5^2+15p_6p_7\neq 0. \tag{9}$$

$$b=-\frac{3p_7}{2p_3+3p_5},\\ c=\frac{1}{p_3+p_5}\left(-p_2+\frac{3p_4p_7}{2p_3+3p_5}\right),\\ \lambda=1+p_4c, p_6=0,\\ p_3+p_5\neq 0, 2p_3+3p_5\neq 0. \tag{10}$$

$$b=-\frac{3p_7}{2p_3+3p_5},\\ p_2=\frac{3p_4p_7}{2p_3+3p_5},\\ \lambda=1+p_4c,\\ p_5=-p_3, p_6=0, p_3\neq 0. \tag{11}$$

$$b=-\frac{p_5}{10p_6},\\ c=\frac{1}{5\gamma p_6^2}\left(-50p_2^2p_6^2+2p_3^2p_4^2+2p_3p_4^2p_5\right.\\ \left.+p_4p_6\gamma+10p_2p_4p_5p_6\right), p_6\neq 0,\\ p_7=\frac{p_5(p_3+p_5)}{15p_6},\\ \lambda=1+p_4c+p_6c^2, 2p_3+p_5\neq 0,\\ -p_3p_4-p_4p_5+5p_2p_6\neq 0. \tag{12}$$

$$b=-\frac{p_5}{10p_6},\\ c=-\frac{p_4}{5p_6},\\ p_2=\frac{p_4(p_3+p_5)}{5p_6},\\ \lambda=1+p_4c+p_6c^2,\\ p_7=\frac{p_5(p_3+p_5)}{15p_6}, 2p_3+p_5\neq 0, p_6\neq 0. \tag{13}$$

Here

$$\alpha = \sqrt{(2p_3+3p_5)^2 - 120p_6p_7},$$
$$\beta = -\frac{p_3p_5}{p_6} - \frac{p_5^2}{p_6} + 15p_7,$$

and

$$\gamma = 10p_2p_3 + 5p_2p_5 - \frac{2p_3^2p_4}{p_6} - \frac{3p_3p_4p_5}{p_6} - \frac{p_4p_5^2}{p_6}.$$

Then we conclude that Eq.(5) has the following solution:

$$u = -\frac{3c}{2b}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-\lambda t-\xi_0)\right]$$

or

$$u = \frac{3c}{2b}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(x-\lambda t-\xi_0)\right]$$

with $c>0$ and the parameters b, c, c_1, c_2, c_3, c_4, c_5, c_6, λ satisfying the relations in Eqs.(7)–(13), respectively. If the parameters in Eqs.(7)–(13) are not particularly specified, it implies that they can be arbitrary.

Case 2 $c_6=0$ in Eq.(5). In this case we set

$$u = a_1v + a_2v^2, \quad v'' = bv^2 + cv, \tag{14}$$

where a_1, a_2, b, c are real constants to be determined. Then we have

$$\begin{aligned}
u'^2 =& \frac{8}{3}a_2^2bv^5 + \left(\frac{8}{3}a_1a_2b + 4a_2^2c\right)v^4 \\
&+ \left(\frac{2}{3}a_1^2b + 4a_1a_2c\right)v^3 + a_1^2cv^2, \\
u'' =& \frac{10}{3}a_2bv^3 + (4a_2c + a_1b)v^2 + a_1cv, \\
u^{(4)} =& \frac{70}{3}a_2b^2v^4 + \frac{10}{3}\left[13a_2bc + a_1b^2\right]v^3 \\
&+ [16a_2c^2 + 5a_1bc]v^2 + a_1c^2v.
\end{aligned} \tag{15}$$

Inserting Eqs.(14) and (15) into Eq.(13) and comparing the coefficients of v with the same powers we obtain that

$$\begin{aligned}
&a_2^2b[5p_5+4p_3] = 0, \\
&a_1c^2p_6 + a_1cp_4 + a_1p_1 = 0, \\
&\frac{70}{3}a_2b^2p_6 + \left[4a_2^2c + \frac{13}{3}a_1a_2b\right]p_5 \\
&+ \left(\frac{8}{3}a_1a_2b + 4a_2^2c\right)p_3 + a_2^2p_2 = 0, \\
&\frac{10}{3}\left(13a_2bc + a_1b^2\right)p_6 + (5a_1a_2c + a_1^2b)p_5 \\
&+ \frac{10}{3}a_2bp_4 + \left(\frac{2}{3}a_1^2b + 4a_1a_2c\right)p_3 + 2a_1a_2p_2 = 0, \\
&(16a_2c^2 + 5a_1bc)p_6 + a_1^2cp_5 + (4a_2c + a_1b)p_4 \\
&+ a_1^2cp_3 + a_1^2p_2 + a_2p_1 = 0,
\end{aligned} \tag{16}$$

where $p_6 = c_5$, $p_5 = c_4$, $p_4 = c_2$, $p_3 = (c_3 - c_4)/2$, $p_2 = c_1/2$ and $p_1 = 1-\lambda$. Solving the algebraic equations (16) we arrive at the following four sets of solutions:

$$\begin{aligned}
a_1 =& -\frac{10bp_6}{2p_2+3p_5}, \\
a_2 =& 0, \\
c =& \frac{10p_2p_6 - 2p_3p_4 - 3p_4p_5}{5p_5p_6}, \\
p_1 =& -cp_4 - c^2p_6;
\end{aligned} \tag{17}$$

$$\begin{aligned}
a_1 =& 0, \quad 5p_5 + 4p_3 = 0, \\
a_2 =& -\frac{910p_6^2}{3(p_4p_5 + 13p_2p_6)}b^2, \\
c =& -\frac{p_4}{13p_6}, \\
p_1 =& \frac{36p_4^2}{169p_6};
\end{aligned} \tag{18}$$

$$\begin{aligned}
p_6 =& 0, \ 4p_3 + 5p_5 = 0, \\
a_1 =& -\frac{2p_4b}{p_2}, \\
a_2 =& \frac{p_4p_5}{p_2^2}b^2, \\
c =& -\frac{p_2}{p_5}, \\
p_1 =& -cp_4;
\end{aligned} \tag{19}$$

$$\begin{aligned}
a_1 =& -\frac{70p_6}{3p_5}b, \\
a_2 =& -\frac{35(12p_2p_6 - p_4p_5)}{27p_2^2}b^2, \\
c =& \frac{p_2}{p_5}, \\
p_1 =& -cp_4 - c^2p_6, \\
& p_4^2p_5^2 - 13p_2p_4p_5p_6 + 15p_2^2p_6^2 = 0.
\end{aligned} \tag{20}$$

Returning to the original coefficients c_i's, we find that Eqs.(17)–(20) generate the following solutions to WWE:

$$u = a_1v = \frac{15cc_5}{c_3+2c_4}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]$$

or,

$$-\frac{15cc_5}{c_3+2c_4}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \tag{21}$$

with $c = \dfrac{5c_1c_5 - 2c_2c_4 - c_2c_3}{5c_4c_5} > 0$, $\lambda = 1 + cc_2 + c^2c_5$;

$$u = a_2v^2 = -\frac{1365c_2^2}{169(2c_2c_4+13c_1c_5)}\mathrm{sech}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right],$$

or

$$-\frac{1365c_2^2}{169(2c_2c_4+13c_1c_5)}\mathrm{csch}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \tag{22}$$

with $2c_2c_4+13c_1c_5 \neq 0$, $c=-\dfrac{c_2}{13c_5}>0$, $\lambda=1-\dfrac{36c_2^2}{169c_5}$ and $2c_3+3c_4=0$;

$$\begin{aligned} u =& a_1v+a_2v^2 \\ =& -\frac{3c_2}{c_4}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right] \\ &+\frac{9c_2}{c_4}\mathrm{sech}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \end{aligned}$$

or

$$\frac{3c_2}{c_4}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]+\frac{9c_2}{c_4}\mathrm{csch}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \quad (23)$$

with $c_5=0$, $2c_3+3c_4=0$, $c=-\dfrac{c_1}{2c_4}>0$, $\lambda=1-\dfrac{c_1c_2}{2c_4}$; and

$$\begin{aligned} u =& a_1v+a_2v^2=\frac{35c_1c_5}{2c_4^2}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right] \\ &-\frac{35(6c_1c_5-c_2c_4)}{12c_4^2}\mathrm{sech}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \end{aligned}$$

or

$$\begin{aligned} &-\frac{35c_1c_5}{2c_4^2}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right] \\ &-\frac{35(6c_1c_5-c_2c_4)}{12c_4^2}\mathrm{csch}^4\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right], \quad (24) \end{aligned}$$

with $2c_3+3c_4=0$, $4c_2^2c_4^2-26c_1c_2c_4c_5+15c_1^2c_5^2=0$, $c=\dfrac{c_1}{2c_4}>0$ and $\lambda=1+cc_2+c^2c_5$.

The exact bell-shaped solitary solution(the first part of (24) here) was obtained by using the ansatz $u(\xi)=My/(1+y)^2, y=\mathrm{e}^{k\xi}$ in Ref.[17].

2.3. Remarks

It should be pointed out that the algebraic equations here are solved by using Mathematica. The generalized fifth order equation (B) can appear when we consider the higher order approximation in the problem described by the KdV equation.[15]

In fact, if the nonlinear term u^5 in Eq.(A) is taken to have a polynomial of degree less than 6, then the same ansatz here applies.

There exist some other nonlinear equations similar to the fifth order KdV, and the second order ordinary differential equation ansatz can be applied to them.

3. Other nonlinear equations

3.1. The second equation in the KdV hierarchy (SEKdV)[15]

$$u_t+\partial_x(u_{4x}+10uu_{xx}+5(u_x)^2+10u^3)=0.$$

Inserting $u(\xi)$ with $\xi=x-\lambda t$ into the above equation and integrating the resultant under the boundary condition $u(-\infty)=0$ we obtain

$$-\lambda u+10u^3+5(u')^2+10uu''+u^{(4)}=0.$$

The same ansatz as that in Eq.(6) will lead to

$$b^2+4b+3=0, \quad bc+3c=0, \quad \lambda=c^2,$$

which give

$$b=-3, \quad \lambda=c^2.$$

Thus

$$u=\frac{c}{2}\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]$$

or

$$-\frac{c}{2}\,\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right],$$

is the wanted solution to the SEKdV equation, where c is an arbitrary positive constant and $\lambda=c^2$.

3.2. Sawada–Kotera equation (SKE)[15]

$$u_t+45u^2u_x+15u_xu_{xx}+15uu_{xxx}+u_{5x}=0.$$

Treating it in the same way as the previous, we obtain its corresponding reduced ordinary differential equation

$$-\lambda u+15u^3+15uu''+u^{(4)}=0.$$

Therefore, by using the same ansatz as that in Eq.(6) the explicit solution to SKE is

$$u=\frac{c}{2}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]$$

or

$$-\frac{c}{2}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right],$$

with c being an arbitrary positive constant and $\lambda=c^2$.

3.3. Kunin equation[15]

$$\begin{aligned} &u_t+u_x+\mu u_{3x}+2quu_x+\beta uu_{3x} \\ &+\delta u_xu_{2x}+3ru^2u_x=0. \end{aligned}$$

The ansatz (6) enables us to obtain its corresponding solution

$$u=\frac{3r\mu-(\delta+2\beta)q}{(\delta+\beta)r}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right]$$

or

$$-\frac{3r\mu-(\delta+2\beta)q}{(\delta+\beta)r}\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi-\xi_0)\right],$$

where $c=\dfrac{2(3r\mu-(\delta+2\beta)q)}{(\delta+\beta)(\delta+2\beta)}>0$, $\lambda=1+\mu c$.

3.4. Benney equation[15]

$$u_t + pu_x + \mu u_{3x} + \alpha u_{5x} - uu_{3x} - 2u_x u_{2x} = 0.$$

The ansatz (6) enables us to obtain its corresponding solution

$$u = \frac{3\mu}{4}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - \xi_0)\right]$$

or

$$-\frac{3\mu}{4}\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - \xi_0)\right],$$

with $c = -\dfrac{\mu}{5\alpha} > 0$, and $\lambda = p + \mu c + \alpha c^2$.

3.5. Kaup equation[15]

$$u_t + u_{5x} + 30uu_{3x} + 75u_x u_{xx} + 180u^2 u_x = 0,$$

whose reduced equation corresponds to

$$-\lambda u + u^{(4)} + 30uu'' + \frac{45}{2}u'^2 + 60u^3 = E,$$

where E is an integration constant. Obviously, we can use the following ansatz:

$$u = A + v, \quad v'' = bv^2 + cv$$

to obtain the solution

$$u = A\left(1 - 3\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(\xi - \xi_0)\right]\right)$$

or

$$A\left(1 + 3\mathrm{csch}^2\left[\frac{\sqrt{c}}{2}(\xi - \xi_0)\right]\right)$$

with $\lambda = c^2 + 30Ac + 180A^2$, $c = -3A$ or $c = -24A$, and $A < 0$, $E = -\lambda A + 60A^3$.

4. Conclusion

We have obtained several solutions to the generalized fifth order nonlinear wave equations. We will further apply these ansatz methods to other nonlinear evolution equations of coupled or reaction-diffusion system in the future study.

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