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Lie-form invariance of the Lagrange system*

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In this paper, the Lie-form invariance of the Lagrange system is studied. The definition and the criterion of the Lie-form invariance of the Lagrange system are given. The Hojman conserved quantity and a new type of conserved quantity deduced from the Lie-form invariance are obtained. Finally, two examples are presented to illustrate the application of the results.

Keywords: analytical mechanics, Lagrange system, symmetry, conserved quantity
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1. Introduction

The research on symmetries and conserved quantities of a mechanical system possesses important theoretical and practical significance. About the symmetry, there are mainly the Noether symmetry,[1−3] the Lie symmetry[2−10] the form invariance,[11−16] etc. And about the conserved quantity, there are mainly the Noether conserved quantity[1−3] the Hojman conserved quantity,[17,18] a new type of conserved quantity[19] and so on. Previous studies on symmetries and conserved quantity are all carried out from a certain kind of symmetry to seek a type of conserved quantity. In this paper, the Lie-form invariance is first defined, from which two kinds of conserved quantities are found.

2. Definition and criterion of the Lie-form invariance

The differential equations of the Lagrange system are

$$E_s(L) = 0 \qquad (s = 1, \ldots, n), \tag{1}$$

where

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s} \tag{2}$$

are the Euler's operators, and $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system. Suppose that the system is not singular, i.e.

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s \partial \dot{q}_k}\right) \neq 0, \tag{3}$$

then from Eqs.(1) all the generalized accelerations can be obtained:

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{4}$$

Definition The symmetry that is both a Lie symmetry and a form invariance is called a Lie-form invariance.

For a Lagrange system, the determining equations of the Lie symmetry are

$$X^{(2)}\{E_s(L)\} = 0. \tag{5}$$

The criterion equations of the form invariance are

$$E_s\left\{X^{(1)}(L)\right\} = 0, \tag{6}$$

where

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s},$$
$$X^{(2)} = X^{(1)} + [(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) - \ddot{q}_s\dot{\xi}_0]\frac{\partial}{\partial \ddot{q}_s}. \tag{7}$$

According to the above definition, the following result can be obtained easily.

Criterion If the generators of infinitesimal transformations ξ_0 and ξ_s satisfy

$$\left\{X^{(2)}[E_s(L)]\right\}^2 + \left\{E_s[X^{(1)}(L)]\right\}^2 = 0, \tag{8}$$

then the corresponding symmetry is a Lie-form invariance of the Lagrange system.

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3. The conserved quantity deduced from the Lie-form invariance

Under some conditions, by using the Lie-form invariance, both the Hojman conserved quantity and a new type of conserved quantity can be obtained.

Proposition 1 Under the special infinitesimal transformations that have unchanging time ($\xi_0 = 0$), if there is a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{9}$$

then the Lie-form invariance will lead to a Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu} \frac{\partial}{\partial q_s}(\mu \xi_s) + \frac{1}{\mu} \frac{\partial}{\partial \dot{q}_s}(\mu \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s) = \text{const.}, \tag{10}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s \frac{\partial}{\partial q_s} + \alpha_s \frac{\partial}{\partial \dot{q}_s}. \tag{11}$$

Proof According to the Definition, a Lie-form invariance must be a Lie symmetry, that is to say, it satisfies

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s = \frac{\partial \alpha_s}{\partial q_k} \xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_k. \tag{12}$$

From expressions (10) and (12), by using Hojman's proof method, Proposition 1 can be proved.

Proposition 2 If there is a function $G_{\mathrm{F}} = G_{\mathrm{F}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\tilde{X}^{(1)}(L) \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 + \tilde{X}^{(1)} \left\{ \tilde{X}^{(1)}(L) \right\} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} G_{\mathrm{F}} = 0, \tag{13}$$

then the Lie-form invariance will lead to a new type of conserved quantity

$$I_{\mathrm{F}} = \tilde{X}^{(1)}(L) \xi_0 + \frac{\partial \tilde{X}^{(1)}(L)}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \xi_0) + G_{\mathrm{F}} = \text{const.} \tag{14}$$

Here

$$\tilde{X}^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s - \dot{q}_s \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 \right) \frac{\partial}{\partial \dot{q}_s}.$$

Proof According to the Definition, a Lie-form invariance must be a form invariance. When a form invariance satisfies condition (13), there is a new type of conserved quantity (14).

4. Illustrative examples

Example 1 The Lagrangian of the Toda crystal lattice with three mass points is

$$\begin{aligned} L =& \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - \exp(q_1 - q_2) \\ & - \exp(q_2 - q_3) - \exp(q_3 - q_1). \end{aligned} \tag{15}$$

Take generators as follows:

$$\xi_0 = 0, \quad \xi_1 = \xi_2 = \xi_3 = 1, \tag{16}$$

then we have

$$E[X^{(1)}(L)] = X^{(2)}[E(L)] = 0. \tag{17}$$

So, generators (16) are Lie-form invariant. Expression (9) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \mu = 0, \tag{18}$$

from which we obtain

$$\mu = 1 \tag{19}$$

and

$$\mu = q_1 + q_2 + q_3 - (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)t. \tag{20}$$

From expressions (16) and (20), by using Proposition 1, the conserved quantity expression (10) yields

$$I_{\mathrm{H}} = 3[q_1 + q_2 + q_3 - (\dot{q}_1 + \dot{q}_2 + \dot{q}_3)t]^{-1} = \text{const.} \tag{21}$$

This is a Hojman conserved quantity of the system deduced from the Lie-form invariance.

Notice that the Noether conserved quantity deduced from generators (16) is

$$I_{\mathrm{N}} = \dot{q}_1 + \dot{q}_2 + \dot{q}_3 = \text{const.} \tag{22}$$

Clearly, it is different from expression (21).

Example 2 The Lagrangian of a system with two degrees of freedom is

$$L = \frac{1}{2} \left[\dot{q}_1^2 + \dot{q}_2^2 f(q_2) \right] - g(q_2), \tag{23}$$

where f and g are both continuously differentiable functions of q_2.

Take generators as follows:

$$\xi_0 = 0, \quad \xi_1 = \dot{q}_1 + t, \quad \xi_2 = 0. \tag{24}$$

It is easy to prove that the corresponding symmetry is Lie-form invariant, and

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_1 =& 1, \\ \tilde{X}^{(1)}(L) =& \dot{q}_1, \\ \tilde{X}^{(1)} \left\{ \tilde{X}^{(1)}(L) \right\} =& 1. \end{aligned} \tag{25}$$

Substituting expressions (24) and (25) into expression (13), we obtain

$$G_{\mathrm{F}} = -t. \tag{26}$$

And expression (14) gives

$$I_{\mathrm{F}} = \dot{q}_1 = \text{const.} \tag{27}$$

This is a new type of conserved quantity of the system deduced from a Lie-form invariance.

Notice that the Noether conserved quantity deduced from generators (24) is

$$I_{\mathrm{N}} = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 t - q_1 = \text{const.} \tag{28}$$

Obviously, it is different from the new type of conserved quantity.

5. Conclusions

The previous studies on symmetries and conserved quantities are all carried out from a kind of symmetry to seek a type of conserved quantity. In this paper, a new kind of symmetry, the Lie-form invariance, is defined. The main result of this paper is that the new symmetry can lead to not only the Hojman conserved quantity but also a new type of conserved quantity.

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