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Form invariances and Lutzky conserved quantities for Lagrange systems*

Mei Feng-Xiang(梅凤翔)[a)†] and Xu Xue-Jun(许学军)[a)b)]

[a)] *Department of Mechanics, Beijing Institute of Technology, Beijing* 100081, *China*

[b)] *Department of Physics, Zhejiang Normal University, Jinhua* 321004, *China*

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This paper focuses on studying a form invariance which can result in a Lutzky conserved quantity for a Lagrange system. The criterion of the form invariance for the system is given. The necessary and sufficient condition under which the form invariance is a Lie symmetry for the system is obtained. Lutzky's result is utilized to prove that the form invariance can lead to a Lutzky conserved quantity. Two examples are finally given to illustrate the application of the result.

Keywords: symmetry, form invariance, conserved quantity, Lagrange system
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1. Introduction

It is well known that a Noether symmetry is an invariance of a Hamilton action under the infinitesimal transformations of group[1−3] and a Lie symmetry is an invariance of differential equations of motion under the infinitesimal transformations of group.[4] For a Lagrange system, if the Lagrangian is replaced by the transformed Lagrangian, whereas the form of the equations of motion is invariant, then the invariance is called a form invariance.[5] Noether conserved quantities can be found by using Noether symmetries.[2,3,6−8] And they can also be obtained by using Lie symmetries[4,7−14] as well as by using form invariances.[5,15−21] Lutzky obtained a few non-Noether conserved quantities from the Lie point symmetries and the velocity-dependent symmetries for Lagrange systems separately.[22−24] In this paper, we will find Lutzky conserved quantities by using the form invariance.

2. Criterion of form invariance for Lagrange systems

Consider a non-singular Lagrange system

$$\mathrm{E}_s(L)=0 \qquad (s=1,2,\cdots,n). \tag{1}$$

From Eqs.(1), one can find all of the generalized accelerations as

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \qquad (s=1,2,\cdots,n). \tag{2}$$

Choose the infinitesimal transformations of group with respect to time and coordinates as

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$$
$$(s=1,2,\cdots,n), \tag{3}$$

where ε is an infinitesimal parameter, and ξ_0,ξ_s are infinitesimal generators. Under the infinitesimal transformations (3), the Lagrangian L becomes L^*

$$L^*=L\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*}\right)=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})+\varepsilon\mathrm{X}^{(1)}(L)+O(\varepsilon^2), \tag{4}$$

where

$$\mathrm{X}^{(1)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}. \tag{5}$$

According to the definition[5] of a form invariance for the Lagrange system, we have

$$\mathrm{E}_s(L^*)=0 \qquad (s=1,2,\cdots,n). \tag{6}$$

Substituting Eq.(4) into Eq.(6), using Eq.(1), and neglecting ε^2 and the higher-order infinitesimal terms,

*Project supported by the National Natural Science Foundation of China (Grant No 10272021).
†E-mail: meifx@bit.edu.cn

we obtain the criterion equations of the form invariance as

$$\mathrm{E}_s\{\mathrm{X}^{(1)}(L)\}=0 \qquad (s=1,2,\cdots,n). \tag{7}$$

3. The necessary and sufficient condition under which the form invariance is a Lie symmetry

Introducing the operator

$$\mathrm{X}^{(2)}=\mathrm{X}^{(1)}+[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\bullet}-\ddot{q}_s\dot{\xi}_0]\frac{\partial}{\partial\ddot{q}_s}, \tag{8}$$

and making the calculation, we have

$$\begin{aligned}&\mathrm{E}_s\{\mathrm{X}^{(1)}(L)\}-\mathrm{X}^{(2)}\{\mathrm{E}_s(L)\}\\ =&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial L}{\partial t}\right)-\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\right)-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}\\ &+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\right)-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\right)+\frac{\partial\dot{\xi}_0}{\partial q_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}\\ &(s=1,2,\cdots,n).\end{aligned} \tag{9}$$

For the Lagrange system, the determining equations of a Lie symmetry have the form

$$\mathrm{X}^{(2)}\{\mathrm{E}_s(L)\}=0 \qquad (s=1,2,\cdots,n). \tag{10}$$

Their equivalent forms are

$$\ddot{\xi}_s-\dot{q}_s\ddot{\xi}_0-2\dot{\xi}_0\alpha_s=\mathrm{X}^{(1)}(\alpha_s) \qquad (s=1,2,\cdots,n). \tag{11}$$

Therefore, we have

Proposition 1 For the Lagrange system, the necessary and sufficient condition under which the form invariance is a Lie symmetry is that the right-hand side of formula (9) vanishes, i.e.

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_0}{\partial\dot{q}_s}\frac{\partial L}{\partial t}\right)-\frac{\partial\xi_0}{\partial q_s}\frac{\partial L}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\right)\\ &-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\right)-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}\\ &-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\xi}_0}{\partial\dot{q}_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\right)+\frac{\partial\dot{\xi}_0}{\partial q_s}\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}-\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}=0\\ &(s=1,2,\cdots,n).\end{aligned} \tag{12}$$

Proof Substituting Eqs.(7) and relation (12) into formula (9), we obtain $\mathrm{X}^{(2)}\{\mathrm{E}_s(L)\}=0$. According to the determining equations (10), we know that the form invariance is a Lie symmetry.

Particularly, if the transformations are point symmetrical, such as

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q}), \quad q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q}) \qquad (s=1,2,\cdots,n), \tag{13}$$

then relation (12) becomes

$$\frac{\partial\xi_0}{\partial q_s}\left[\frac{\partial L}{\partial t}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}_k\frac{\partial L}{\partial\dot{q}_k}\right)\right]+\ddot{\xi}_0\frac{\partial L}{\partial\dot{q}_s}=0 \qquad (s=1,2,\cdots,n). \tag{14}$$

If $\xi_0=0$, then the relation (12) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\xi_k}{\partial\dot{q}_s}\frac{\partial L}{\partial q_k}\right)-\frac{\partial\xi_k}{\partial q_s}\frac{\partial L}{\partial q_k}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\dot{\xi}_k}{\partial\dot{q}_s}\frac{\partial L}{\partial\dot{q}_k}\right)\\ &-\frac{\partial\dot{\xi}_k}{\partial q_s}\frac{\partial L}{\partial\dot{q}_k}=0 \qquad (s=1,2,\cdots,n).\end{aligned} \tag{15}$$

4. Lutzky conserved quantity deduced from form invariance

Under the transformations (13), if the infinitesimal generators ξ_0,ξ_s satisfy Eq.(10) or (11), then there exists a conserved quantity for the Lagrange system[22]

$$\Phi=2\left(\frac{\partial\xi_s}{\partial q_s}-\dot{q}_s\frac{\partial\xi_0}{\partial q_s}\right)-n\dot{\xi}_0+\mathrm{X}^{(1)}(\ln D). \tag{16}$$

The result has been generalized to non-conservative systems by Fu and Chen.[25] Under the transformations (3) and $\xi_0=0$, if the infinitesimal generators ξ_s satisfy Eq.(10) or (11), then there exists a conserved quantity for the system[23,24]

$$\Phi=\frac{\partial\xi_s}{\partial q_s}+\frac{\partial\dot{\xi}_s}{\partial\dot{q}_s}+\mathrm{X}^{(1)}(\ln D), \tag{17}$$

where

$$D=\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\right). \tag{18}$$

We call the conserved quantity (16) as well as (17) a Lutzky conserved quantity.

Using the form invariance, we can find the Lutzky conserved quantity. We have

Proposition 2 For the Lagrange system, under the transformations (13), if the infinitesimal generator $\xi_0=\xi_0(t,\boldsymbol{q})$ satisfies the relation (14), then the form invariance will lead to the Lutzky conserved quantity (16).

Proof If the infinitesimal generator ξ_0 satisfies relation (14), using Proposition 1, we know that the

generators ξ_0, ξ_s are Lie symmetrical. And we can subsequently obtain the conserved quantity (16) by using the Lutzky's result.[22]

Proposition 3 For the Lagrange system, under the transformations (3) and $\xi_0 = 0$, if the infinitesimal generators $\xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy relation (15), then the form invariance will lead to the Lutzky conserved quantity (17).

Proof If the infinitesimal generators ξ_s satisfy relation (15), using Proposition 1, we know that the generators ξ_s are Lie symmetrical. And the conserved quantity (17) can be gained from the Lutzky's result.[23,24]

5. Illustrative examples

Example 1 The Lagrangian of a system with two degrees of freedom is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - q_2. \tag{19}$$

Choose the infinitesimal generators as

$$\xi_0 = 0, \quad \xi_1 = q_2 + \frac{1}{2}t^2, \quad \xi_2 = t. \tag{20}$$

They are form invariant and satisfy relation (14). Proposition 2 gives the Lutzky conserved quantity

$$\Phi = 0, \tag{21}$$

and it is trivial. Choose the infinitesimal generators as

$$\xi_0 = \xi_2 = 0, \quad \xi_1 = (q_1 - \dot{q}_1 t)\left(q_2 - \dot{q}_2 t - \frac{1}{2}t^2\right). \tag{22}$$

They are form invariant and satisfy relation (15). Proposition 3 gives the Lutzky conserved quantity

$$\Phi = q_2 - \dot{q}_2 t - \frac{1}{2}t^2 = \text{const.} \tag{23}$$

Example 2 Consider a system of two degrees of freedom with Lagrangian

$$\begin{aligned} L =& \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{b}\dot{q}_1 \mathrm{tg}^{-1}(bt) \\ &+ \frac{1}{2b}\dot{q}_2 \ln(1 + b^2t^2) \qquad (b = \text{const}). \end{aligned} \tag{24}$$

Choose the infinitesimal generators as

$$\begin{aligned} \xi_0 =& \xi_2 = 0, \\ \xi_1 =& \left\{ q_1 + \frac{1}{b}\int \mathrm{tg}^{-1}(bt)\mathrm{d}t - t\left[\dot{q}_1 + \frac{1}{b}\mathrm{tg}^{-1}(bt)\right]\right\} \\ &\times \left\{ q_2 + \frac{1}{2b}\int \ln(1 + b^2t^2)\mathrm{d}t \right. \\ &\left. - t\left[\dot{q}_2 + \frac{1}{2b}\ln(1 + b^2t^2)\right]\right\}, \end{aligned} \tag{25}$$

and Proposition 3 gives the Lutzky conserved quantity

$$\begin{aligned} \Phi =& q_2 + \frac{1}{2b}\int \ln(1 + b^2t^2)\mathrm{d}t \\ &- t\left[q_2 + \frac{1}{2b}\ln(1 + b^2t^2)\right] = \text{const.} \end{aligned} \tag{26}$$

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