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Parametric optimum design of an irreversible spin quantum refrigeration cycle*

Lin Bi-Hong(林比宏)[a)b)] and Chen Jin-Can(陈金灿)[a)†]

[a)] *Department of Physics, Xiamen University, Xiamen 361005, China*

[b)] *Department of Physics, Quanzhou Normal College, Quanzhou 362000, China*

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The general performance characteristics of an irreversible quantum refrigeration cycle using many non-interacting spin-1/2 systems as the working substance and consisting of two adiabatic and two isomagnetic field processes are investigated, based on the quantum master equation and semi-group approach. Expressions for several important performance parameters such as the coefficient of performance, cooling rate and power input are derived. By using numerical solutions, the cooling rate of the refrigeration cycle subject to the finite cycle duration is optimized. The maximum cooling rate and the corresponding parameters are calculated numerically. The optimal region of the coefficient of performance and the optimal ranges of the temperatures of the working substance and the times spent on the two isomagnetic field processes are determined. Moreover, the optimal performance of the cycle in the high-temperature limit is also analysed in detail. The results obtained here are further generalized, so that they may be directly used to describe the performance of the quantum refrigeration cycle using spin-J systems as the working substance.

Keywords: quantum refrigeration cycles, quantum master equation, parametric optimum design
PACC: 0720M, 0365, 0570

1. Introduction

Recently, the investigation relative to quantum thermodynamic cycles has attracted a good deal of attention. The quantum master equation and semi-group approach have been successfully used to investigate the optimal performance of the quantum thermodynamic cycles working with spin systems[1−6] and harmonic oscillator systems,[7−11] and some novel results have been obtained. The investigation on the optimal performance of quantum thermodynamic cycles will be helpful to understand deeply the relationship and distinguish between the quantum and the classical thermodynamic cycles.

Like quantum thermodynamic cycles working with harmonic oscillators,[7−12] quantum thermodynamic cycles using spin systems as the working substance may have three typical cycle models. They are the quantum Carnot cycle[4,6,13] consisting of two isothermal and adiabatic (i.e. constant-polarization) processes, Ericsson cycle[1,2] consisting of two isothermal and two isomagnetic field processes, and Brayton cycle[3] consisting of two adiabatic (i.e. constant-polarization) and two isomagnetic field processes. In the last years, many authors have concentrated on the investigation of the optimal performance of the spin quantum Carnot cycle and Ericsson cycle, and deduced some important fundamental optimal relations and performance bounds of the cycles. Also, Feldmann and Kosloff have investigated the optimal performance of the quantum Brayton heat engine[3,5] and heat pump[3] working with spin-1/2 systems by using the dynamical model which is based on the probability distribution of occupancy of the energy levels evolving and a Pauli master equation. However, the performance characteristics of a quantum Brayton refrigeration cycle working with spin systems, which is simply referred to as a spin quantum Brayton refrigeration cycle, has been rarely investigated.

In this article, we mainly investigate the performance of an irreversible spin quantum Brayton refrigeration cycle consisting of two adiabatic and two isomagnetic field processes. General expressions for

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†Author to whom all correspondence should be addressed. E-mail: jcchen@xmu.edu.cn

the coefficient of performance, cooling rate, power input of the cycle are derived, based on the quantum master equation and the dynamical semi-group approach of the quantum theory of open systems. The important performance parameters of the cycle are optimized and the general performance characteristics of the cycle are analysed. The curves of the fundamental optimum relation between the cooling rate and the coefficient of performance are presented. The optimally operating regions of some performance parameters in the cycle are determined.

2. A quantum refrigeration cycle

First of all, we consider a quantum refrigeration cycle operating between two heat reservoirs at constant temperatures $T_{\rm h}$ and $T_{\rm c}$. The working substance of the cycle consists of many non-interacting spin-1/2 systems. In the quantum refrigeration cycle, the spin systems are not only coupled mechanically to a given "magnetic field" $\omega(t)$, but are also coupled to a heat reservoir at a temperature of T. Based on the quantum mechanics, the Hamiltonian of the interaction between a magnetic moment $\boldsymbol{M}$ and a magnetic field $\boldsymbol{B}$ is given by $\hat{\boldsymbol{H}}(t) = -\hat{\boldsymbol{M}} \cdot \boldsymbol{B}$, where the magnitude of the magnetic field can change over time, but is not allowed to reach zero. For a single-spin quantum system, the magnetic moment $\boldsymbol{M}$ is proportional to the spin angular momentum $\boldsymbol{S}$. When the direction of the magnetic field $\boldsymbol{B}$ is chosen to be constant and along the positive z-axis, the Hamiltonian may be given by $\hat{\boldsymbol{H}}(t) = 2\mu_{\rm B}\hat{\boldsymbol{S}} \cdot \boldsymbol{B} = 2\mu_{\rm B}B_z(t)\hat{\boldsymbol{S}}_z$, where $\mu_{\rm B}$ is the Bohr magnetron. As described in Ref.[6], one can refer to $\omega\,(t) = 2\mu_{\rm B}B_z\,(t)$ rather than B_z as "the field" for simplicity. It is clear that ω is positive for the spin angular momentum and magnetic moment are in opposite directions. Therefore, the Hamiltonian of an isolated single spin-1/2 system in the presence of the field $\omega(t)$ may be expressed as

$$\hat{\boldsymbol{H}}(t) = \omega(t)\hat{\boldsymbol{S}}_z, \tag{1}$$

and the internal energy of the spin system may be given by

$$E = \langle \hat{\boldsymbol{H}} \rangle = \omega(t)\langle \hat{\boldsymbol{S}}_z \rangle = \omega S. \tag{2}$$

According to statistical mechanics, the expectation value of a spin-1/2 angular momentum $\hat{S}_z$ is given by

$$\langle \hat{\boldsymbol{S}}_z \rangle = S = -\frac{1}{2}\tanh(\beta'\omega/2), \tag{3}$$

where $\beta' = 1/T'$ (T' is the temperature of the spin quantum system in energy units) and $-1/2 < S < 0$ (our units are such that $\hbar = 1$). For the convenience of writing, "temperature" will refer to β' (or β) rather than T' (or T).

Using Eq.(3), one can plot the cycle diagram of a spin quantum refrigeration cycle consisting of two constant-polarization and two isomagnetic field processes, as shown in Fig.1. It can be seen from Fig.1 that in the cycle, the two adiabatic processes (i.e. constant-polarization) are connected by the two isomagnetic field processes $\omega = \omega_1$ and $\omega = \omega_2$ with $\omega_2 > \omega_1$. The two isomagnetic field processes in the cycle correspond to the two constant-pressure processes in a gas Brayton refrigeration cycle, while the two adiabatic processes in the cycle are identical to the two processes that the polarization of the spin system $S = S_1$ and S_2 are kept constant.[6] Therefore, the quantum refrigeration cycle shown in Fig.1 may be referred to as the spin quantum Brayton refrigeration cycle.

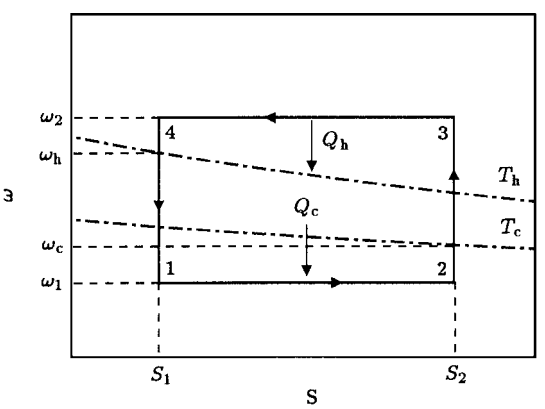


Fig.1. The S-ω diagram of an irreversible quantum refrigeration cycle, where ω is in Joule units.

In the two isomagnetic field processes, the spin systems are, respectively, coupled to the heat reservoirs at constant temperatures $\beta = \beta_{\rm h}$ and $\beta = \beta_{\rm c}$, and the amounts of heat exchange between the working substance and the heat reservoirs are represented by $Q_{\rm h}$ and $Q_{\rm c}$. Due to the finite-rate heat transfer between the working substance and the heat reservoirs, the temperatures β_1, β_2, β_3, and β_4 of the working substance in states 1, 2, 3, and 4 are different from those of the heat reservoirs and there exists a relation: $\beta_{\rm c} \geq \beta_1 > \beta_2 > \beta_4 > \beta_3 \geq \beta_{\rm h}$. Using Eq.(3) and Fig.1, one can obtain the following relations:

$$\beta_3\omega_2 = \beta_2\omega_1 \tag{4}$$

and

$$\beta_4\omega_2 = \beta_1\omega_1. \tag{5}$$

It is thus clear that, for a spin quantum Brayton refrigeration cycle, there is an important relation, $\beta_1/\beta_2 = \beta_4/\beta_3$, which restricts the temperatures of the four states 1, 2, 3, and 4 in Fig.1. It is of interest to note that the important relation is identical with that of a Brayton refrigeration cycle working with an ideal gas.

3. Several important performance parameters

The coefficient of performance, cooling rate, and power input are three important performance parameters, which are often considered in the optimal design and theoretical analysis of the refrigerators. In order to obtain the general expressions of the parameters, the times of the heat-exchange processes have to be calculated. To this end, we begin to solve the equation of motion that determines the time evolution of the spin angular momentum.

Based on the semi-group formalism, the equation of motion of an operator $\hat{\boldsymbol{X}}$ in the Heisenberg picture is given by the quantum master equation,[6,9,14−16] i.e.

$$\frac{\mathrm{d}\hat{\boldsymbol{X}}}{\mathrm{d}t} = \mathrm{i}[\hat{\boldsymbol{H}}, \hat{\boldsymbol{X}}] + \frac{\partial \hat{\boldsymbol{X}}}{\partial t} + L_{\mathrm{D}}(\hat{\boldsymbol{X}}), \tag{6}$$

where $L_{\mathrm{D}}(\hat{\boldsymbol{X}}) = \sum_{\alpha} \gamma_{\alpha}(\hat{\boldsymbol{V}}_{\alpha}^{+}[\hat{\boldsymbol{X}}, \hat{\boldsymbol{V}}_{\alpha}] + [\hat{\boldsymbol{V}}_{\alpha}^{+}, \hat{\boldsymbol{X}}]\hat{\boldsymbol{V}}_{\alpha})$ is a dissipation term and originates from a thermal coupling of the spin to a heat reservoir, $\boldsymbol{V}_{\alpha}^{+}$ and $\boldsymbol{V}_{\alpha}$ are operators in the Hilbert space of the system and Hermitian conjugates, and γ_{α} are phenomenological positive coefficients. For a spin system, $\boldsymbol{V}_{\alpha}^{+}$ and $\boldsymbol{V}_{\alpha}$ may be chosen to be the spin annihilation and creation operators: $\hat{\boldsymbol{S}}_{-} = \hat{\boldsymbol{S}}_x - \mathrm{i}\hat{\boldsymbol{S}}_y$ and $\hat{\boldsymbol{S}}_{+} = \hat{\boldsymbol{S}}_x + \mathrm{i}\hat{\boldsymbol{S}}_y$, and $\hat{\boldsymbol{H}} = \omega\hat{\boldsymbol{S}}_z$. Substituting $\hat{\boldsymbol{S}}_{-}$, $\hat{\boldsymbol{S}}_{+}$, $\hat{\boldsymbol{H}}$, and $\hat{\boldsymbol{X}} = \hat{\boldsymbol{S}}_z$ into Eq.(6), one can prove[6] that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -a\mathrm{e}^{q\beta\omega}[2(1+\mathrm{e}^{\beta\omega})S + \mathrm{e}^{\beta\omega} - 1], \tag{7}$$

where a and q are constant parameters which describe the interaction between the spin system and the heat reservoirs and their reasonable ranges are $a > 0$ and $-1 < q < 0$; β, ω, and S are, in general, dependent on time.[6] The solution of Eq.(7) is given by

$$t = -\frac{1}{a}\int_{S_{\mathrm{i}}}^{S_{\mathrm{f}}} \frac{\mathrm{d}S}{\mathrm{e}^{q\beta\omega}[2(1+\mathrm{e}^{\beta\omega})S + \mathrm{e}^{\beta\omega} - 1]}, \tag{8}$$

where S_{i} and S_{f} are the initial and final values of S along a given path $S(\beta', \omega)$. Equation (8) is a general expression of time evolution for a spin-1/2 system coupling with the heat reservoir and the external magnetic field.

Using Eqs.(3) and (8), one can calculate the times spent on the previous processes in the cycle. Substituting $S_{\mathrm{i}} = S_1$, $S_{\mathrm{f}} = S_2$, $\beta = \beta_{\mathrm{c}}$ and $\omega = \omega_1$ into Eq.(8), one obtains the time of the isomagnetic field process with $\omega = \omega_1$ as

$$t_{12} = \frac{1}{2a\mathrm{e}^{q\beta_{\mathrm{c}}\omega_1}(1+\mathrm{e}^{\beta_{\mathrm{c}}\omega_1})} \times \ln\left[\frac{2(1+\mathrm{e}^{\beta_{\mathrm{c}}\omega_1})S_1 + \mathrm{e}^{\beta_{\mathrm{c}}\omega_1} - 1}{2(1+\mathrm{e}^{\beta_{\mathrm{c}}\omega_1})S_2 + \mathrm{e}^{\beta_{\mathrm{c}}\omega_1} - 1}\right]. \tag{9}$$

Similarly, substituting $S_{\mathrm{i}} = S_2$, $S_f = S_1$, $\beta = \beta_{\mathrm{h}}$ and $\omega = \omega_2$ into Eq.(8), one obtains the time of the isomagnetic field process with $\omega = \omega_2$ as

$$t_{34} = \frac{1}{2a\mathrm{e}^{q\beta_{\mathrm{h}}\omega_2}(1+\mathrm{e}^{\beta_{\mathrm{h}}\omega_2})} \times \ln\left[\frac{2(1+\mathrm{e}^{\beta_{\mathrm{h}}\omega_2})S_2 + \mathrm{e}^{\beta_{\mathrm{h}}\omega_2} - 1}{2(1+\mathrm{e}^{\beta_{\mathrm{h}}\omega_2})S_1 + \mathrm{e}^{\beta_{\mathrm{h}}\omega_2} - 1}\right]. \tag{10}$$

Before considering further the times spent on the two adiabatic processes, we first derive the first law of thermodynamics for a spin system. Replacing $\hat{\boldsymbol{X}}$ by $\hat{H}$ in Eq.(6) and using Eq.(2), one can obtain the rate of change of the internal energy as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{H}\rangle = \left\langle\frac{\partial\hat{H}}{\partial t}\right\rangle + \langle L_{\mathrm{D}}(\hat{H})\rangle = S\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega\frac{\mathrm{d}S}{\mathrm{d}t}. \tag{11}$$

Equation (11) shows that the internal energy change of the working substance can occur either by transferring population from one level to the other (changing S) or by changing the energy gap between the two levels (changing ω). The population transfer is the microscopic realization of heat exchange, while the energy change due to external field variation is associated with work. Consequently, Eq.(11) is the time derivative of the first law of thermodynamics, and the inexact differentials of the work and heat are, respectively, given by[6,9,16−18]

$$\mathrm{d}W = S\mathrm{d}\omega \tag{12}$$

and

$$\mathrm{d}Q = \omega\mathrm{d}S. \tag{13}$$

It can be seen from Eq.(13) that there exist two adiabatic processes: $\mathrm{d}Q = 0$ and $\mathrm{d}S = 0$. Consequently, the times spent on the two adiabatic processes are negligible[6] compared with those spent on the isomagnetic field processes, and the cycle period is given by

$$t = t_{12} + t_{34}. \tag{14}$$

Using Eq.(13), one can find that the amounts of heat exchange between the working substance and the heat reservoirs in the two isomagnetic field processes mentioned above are, respectively, given by

$$Q_{\mathrm{c}} = Q_{12} = \int_{S_1}^{S_2} \omega_1 \mathrm{d}S = \omega_1(S_2 - S_1) \tag{15}$$

and

$$Q_{\mathrm{h}} = Q_{34} = \int_{S_2}^{S_1} \omega_2 \mathrm{d}S = \omega_2(S_1 - S_2). \tag{16}$$

The work input per cycle can be calculated from Eqs.(15) and (16) as

$$W = |Q_{\mathrm{h}} + Q_{\mathrm{c}}| = (\omega_2 - \omega_1)(S_2 - S_1). \tag{17}$$

From Eqs.(9), (10), and (14)–(17), one can derive the coefficient of performance, cooling rate, and power input of a spin quantum Brayton refrigeration cycle. They can be, respectively, expressed as

$$\begin{aligned}\varepsilon =& \frac{Q_{\mathrm{c}}}{W} = \frac{\omega_1}{\omega_2 - \omega_1} \\ =& \frac{\beta_4 \tanh^{-1}(2S_2)}{\beta_2 \tanh^{-1}(2S_1) - \beta_4 \tanh^{-1}(2S_2)},\end{aligned} \tag{18}$$

$$\begin{aligned}R =& \frac{Q_{\mathrm{c}}}{t} = 2a\omega_1(S_2 - S_1) \\ & \times \left[\frac{1}{\mathrm{e}^{q\beta_{\mathrm{c}}\omega_1} C_1} \ln\left(\frac{2C_1S_1 + C_2}{2C_1S_2 + C_2}\right)\right. \\ & \left. + \frac{1}{\mathrm{e}^{q\beta_{\mathrm{h}}\omega_2} D_1} \ln\left(\frac{2D_1S_2 + D_2}{2D_1S_1 + D_2}\right)\right]^{-1},\end{aligned} \tag{19}$$

and

$$\begin{aligned}P =& \frac{W}{t} = 2a(\omega_2 - \omega_1)(S_2 - S_1) \\ & \times \left[\frac{1}{\mathrm{e}^{q\beta_{\mathrm{c}}\omega_1} C_1} \ln\left(\frac{2C_1S_1 + C_2}{2C_1S_2 + C_2}\right)\right. \\ & \left. + \frac{1}{\mathrm{e}^{q\beta_{\mathrm{h}}\omega_2} D_1} \ln\left(\frac{2D_1S_2 + D_2}{2D_1S_1 + D_2}\right)\right]^{-1},\end{aligned} \tag{20}$$

where $C_1 = 1 + \mathrm{e}^{\beta_{\mathrm{c}}\omega_1}$, $C_2 = \mathrm{e}^{\beta_{\mathrm{c}}\omega_1} - 1$, $D_1 = 1 + \mathrm{e}^{\beta_{\mathrm{h}}\omega_2}$, and $D_2 = \mathrm{e}^{\beta_{\mathrm{h}}\omega_2} - 1$. On the basis of the expressions of these parameters, one can discuss the optimal performance of an irreversible spin quantum Brayton refrigeration cycle using many non-interacting spin-1/2 systems as the working substance.

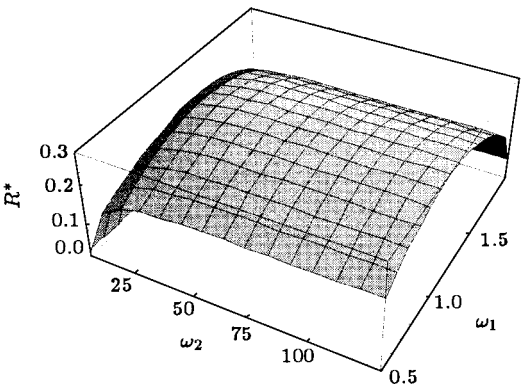


Fig.2. The dimensionless cooling rate of the cycle as a function of the fields (ω_1, ω_2). The graph is presented for the parameters kT_{h}=4.0J, kT_{c}=2.0J, ω_{h}=6.0J, ω_{c}=2.0J, $q = -0.5$ and $\hbar = 1$.

4. Optimal performance characteristics

Using Eq.(19), one can plot a three-dimensional diagram (ω_1, ω_2, R^*) for given q, β_{c}, β_{h}, ω_{h}, and ω_{c}(ω_{h} is the lower bound of ω in the high-isomagnetic field processes and ω_{c} is the upper bound of ω in the low-isomagnetic field processes), as shown in Fig.2, where $R^* = R/(2a\omega_{\mathrm{c}})$ is the dimensionless cooling rate. It can be seen from Fig.2 that there is a maximum for the cooling rate. According to Eq.(19) and the extreme condition $\partial R^*/\partial\omega_1 = 0$, we can obtain the following equation:

$$\begin{aligned}&\frac{t_{12} + t_{34}}{\beta_{\mathrm{c}} C_1^{-1} \mathrm{e}^{-q\beta_{\mathrm{c}}\omega_1}} - \omega_1 \left\{ \frac{4\beta_{\mathrm{c}}(S_2 - S_1)\mathrm{e}^{\beta_{\mathrm{c}}\omega_1}}{(2C_1S_1 + C_2)(2C_1S_2 + C_2)}\right. \\ &\left. - \frac{[(q+1)\mathrm{e}^{\beta_{\mathrm{c}}\omega_1} + q]}{C_1} \ln\left(\frac{2C_1S_1 + C_2}{2C_1S_2 + C_2}\right)\right\} = 0.\end{aligned} \tag{21}$$

Equation (21) gives an optimal relation between $\beta_2(\omega_1)$ and $\beta_4(\omega_2)$ for given q, β_{c}, β_{h}, ω_{h}, and ω_{c}, but it is too complicated to yield a simple analytical solution. However, for given q, β_{h}, β_{c}, ω_{h}, and ω_{c}, the optimal curves between the cooling rate and coefficient of performance R^*-ε and the other optimal characteristic curves (P^*-ε, R^*-ω_1, P^*-ω_1, and ω_j/ω_i-ε) can be plotted by using Eqs.(18)–(21), as shown in Figs.3–5, respectively, where $P^* = P/(2a\omega_{\mathrm{c}})$ is the dimensionless power input. In these figures, the parameters kT_{h}=4.0J, kT_{c}=2.0J, ω_{h}=6.0J, ω_{c}=2.0J, and $q = -0.3$ or $q = -0.5$ are adopted.[6]

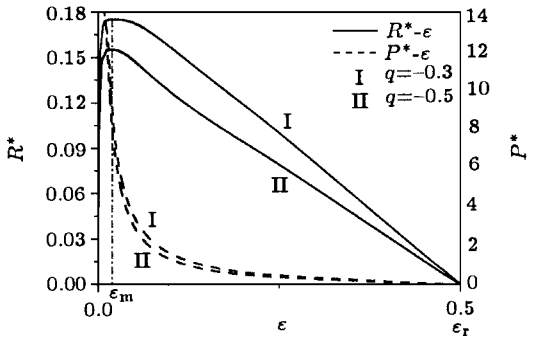


Fig.3. The dimensionless cooling rate $R^* = R/(2a\omega_c)$ (solid lines) and power input $P^* = P/(2a\omega_c)$ (dashed lines) versus the coefficient of performance ε. Curves I and II are presented for $q = -0.3$ and $q = -0.5$, respectively. The values of the parameters kT_c, kT_h, ω_h, ω_c, and $\hbar$ are the same as those used in Fig.2.

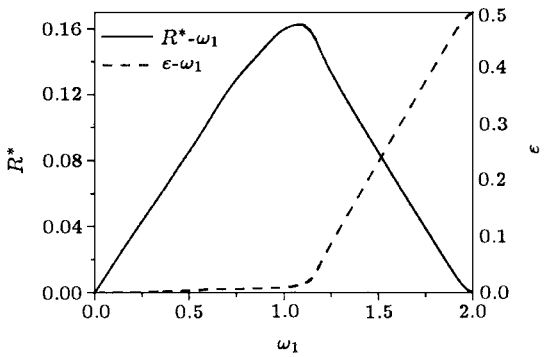


Fig.4. Dimensionless cooling rate R^* (solid line) and coefficient of performance ε (dashed line) versus the field ω_1. The values of the parameters kT_c, kT_h, ω_h, ω_c, q and $\hbar$ are the same as those used in Fig.2.

It is seen from the curves in Fig.3 that there exist a maximum cooling rate R_{max} and a corresponding coefficient of performance ε_m for a set of given parameters q, β_h, β_c, ω_h, and ω_c. Obviously, for different given parameters, the maximum cooling rate R_{max} and corresponding coefficient of performance ε_m will be different. For example, for a given ω_c/ω_h, the larger the temperature ratio T_c/T_h of the two heat reservoirs is, the larger the maximum cooling rate while the smaller the corresponding coefficient of performance will be; for a given T_c/T_h, the smaller the field ratio ω_c/ω_h of the external magnetic field is, the larger the maximum cooling rate and the smaller the corresponding coefficient of performance will be, as indicted in Table 1.

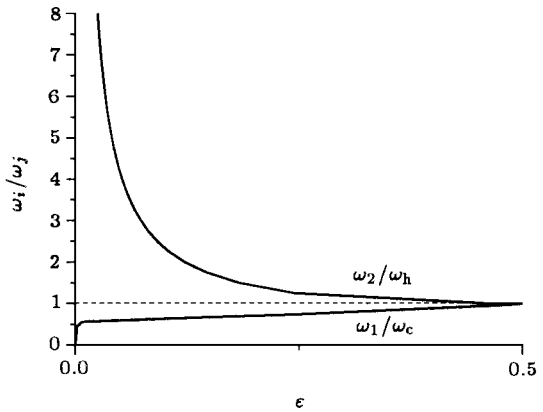


Fig.5. The ω_1/ω_c-ε and ω_2/ω_h-ε characteristic curves. The values of the parameters kT_c, kT_h, ω_h, ω_c, q, and $\hbar$ are the same as those used in Fig.2.

It is also seen from the solid curves in Fig.3 that when $R < R_{max}$, there are two different coefficients of performance for a given cooling rate R: one is smaller than ε_m, and the other is larger than ε_m. When $\varepsilon < \varepsilon_m$, the cooling rate decreases as the coefficient of performance decreases. It is thus clear that the region of $\varepsilon < \varepsilon_m$ is not optimal for a spin quantum Brayton refrigeration cycle. Consequently, the optimal region of the coefficient of performance should be

$$\varepsilon_m \le \varepsilon < \varepsilon_r, \tag{22}$$

where $\varepsilon_r = \omega_c/(\omega_h - \omega_c)$ is the maximum coefficient of performance of a spin Brayton refrigeration cycle. When a quantum Brayton refrigeration cycle is operated in this region, the cooling rate will increase as the coefficient of performance decreases, and vice versa. This indicates that ε_m is an important parameter for a spin quantum Brayton refrigeration cycle. It determines the allowable value of the lower bound of the optimal coefficient of performance.

Table 1. Optimal parameters at the maximum cooling rate for given T_c/T_h, ω_c/ω_h and $q = -0.5$.

T_c/T_h	ω_c/ω_h	R^*_{max}	ε_m	P_m	T_{2m}/T_{4m}
0.48	0.3	0.1623	0.01249	12.998	0.0191
	0.4	0.1309	0.01514	8.642	0.0179
0.50	0.3	0.1668	0.01211	13.777	0.0200
	0.4	0.1354	0.01476	9.170	0.0181
0.52	0.3	0.1692	0.01186	14.103	0.0201
	0.4	0.1365	0.01456	9.531	0.0186

Using Eq.(22) and the above results, we can further find that the optimal values of the power input should be

$$P \leq P_{\rm m} \tag{23}$$

and that the optimal ranges of the "temperature" of the working substance in states 2 and 4 in the two isomagnetic field processes should be

$$\beta_{2\rm m} \geq \beta_2 > \beta_{\rm c} \tag{24}$$

and

$$\beta_{4\rm m} \leq \beta_4 < \beta_{\rm h}, \tag{25}$$

where the values of $P_{\rm m}$, $\beta_{2\rm m}$, and $\beta_{4\rm m}$ can be calculated from Eqs.(19)–(21) and have been listed in Table 1. From Eqs.(4), (5), (24) and (25), we can obtain, respectively, the optimal ranges of the "temperature" of the working substance in states 1 and 3 as

$$\beta_{1\rm m} = \beta_{4\rm m}\omega_{2\rm m}/\omega_{1\rm m} \leq \beta_1 < \beta_{\rm h}\omega_2/\omega_1, \tag{26}$$

and

$$\beta_{3\rm m} = \beta_{2\rm m}\omega_{1\rm m}/\omega_{2\rm m} \geq \beta_3 > \beta_{\rm c}\omega_1/\omega_2. \tag{27}$$

Obviously, Eqs.(23)–(27) provide several significant criteria for the selection of optimally operating conditions. Thus, $P_{\rm m}$, $\beta_{1\rm m}$, $\beta_{2\rm m}$, $\beta_{3\rm m}$ and $\beta_{4\rm m}$ are also important performance parameters for a spin quantum Brayton refrigeration cycle.

In addition, using Eqs.(9), (10), (21), and (22), one can find that the times spent on the two isomagnetic field processes, t_{12} and t_{34}, of the cycle should be controlled to satisfy the following conditions:

$$t_{12} \geq (t_{12})_{\rm m} \tag{28}$$

and

$$t_{34} \geq (t_{34})_{\rm m}, \tag{29}$$

where

$$(t_{12})_{\rm m} = \frac{1}{2a{\rm e}^{q\beta_{\rm c}\omega_{1\rm m}}(1+{\rm e}^{\beta_{\rm c}\omega_{1\rm m}})} \times \ln\left[\frac{2(1+{\rm e}^{\beta_{\rm c}\omega_{1\rm m}})S_1+{\rm e}^{\beta_{\rm c}\omega_{1\rm m}}-1}{2(1+{\rm e}^{\beta_{\rm c}\omega_{1\rm m}})S_2+{\rm e}^{\beta_{\rm c}\omega_{1\rm m}}-1}\right]$$

and

$$(t_{34})_{\rm m} = \frac{1}{2a{\rm e}^{q\beta_{\rm h}\omega_{2\rm m}}(1+{\rm e}^{\beta_{\rm h}\omega_{2\rm m}})} \times \ln\left[\frac{2(1+{\rm e}^{\beta_{\rm h}\omega_{2\rm m}})S_2+{\rm e}^{\beta_{\rm h}\omega_{2\rm m}}-1}{2(1+{\rm e}^{\beta_{\rm h}\omega_{2\rm m}})S_1+{\rm e}^{\beta_{\rm h}\omega_{2\rm m}}-1}\right].$$

If not, the quantum heat engine could not be operated in the rational region.

Figures 3–5 show that the power input P and the field ratio $\omega_2/\omega_{\rm h}$ are of monotonically decreasing functions of the coefficient of performance ε, while the field ratio $\omega_1/\omega_{\rm c}$ is a monotonically increasing function of the coefficient of performance ε. When $\beta_2 = \beta_{\rm c}$ and $\beta_4 = \beta_{\rm h}$, one has $\varepsilon = \varepsilon_{\rm r}$, $R = 0$, and $P = 0$. In such a case, the refrigeration cycle attains its maximum coefficient of performance, but its cooling rate is equal to zero so that it loses its refrigeration role.

It is worthwhile to point out that when the temperatures of two heat reservoirs are high enough, i.e. $\beta\omega \ll 1$, the results obtained above can be simplified considerably. For example, Eqs.(3), (14), and (18)–(21) can be, respectively, written simply as

$$S = -\frac{1}{4}\beta'\omega, \tag{30}$$

$$t = \frac{1}{4a}\ln\left[\frac{(4S_1+\beta_{\rm c}\omega_1)(4S_2+\beta_{\rm h}\omega_2)}{(4S_2+\beta_{\rm c}\omega_1)(4S_1+\beta_{\rm h}\omega_2)}\right], \tag{31}$$

$$\varepsilon = \frac{Q_{\rm c}}{W} = \frac{\omega_1}{\omega_2-\omega_1} = \frac{S_2\beta_4}{S_1\beta_2 - S_2\beta_4}, \tag{32}$$

$$R = \frac{Q_{\rm c}}{t} = 4a\omega_1(S_2-S_1) \times \ln\left[\frac{(4S_1+\beta_{\rm c}\omega_1)(4S_2+\beta_{\rm h}\omega_2)}{(4S_2+\beta_{\rm c}\omega_1)(4S_1+\beta_{\rm h}\omega_2)}\right]^{-1}, \tag{33}$$

$$P = \frac{W}{t} = 4a(\omega_2-\omega_1)(S_2-S_1) \times \ln\left[\frac{(4S_1+\beta_{\rm c}\omega_1)(4S_2+\beta_{\rm h}\omega_2)}{(4S_2+\beta_{\rm c}\omega_1)(4S_1+\beta_{\rm h}\omega_2)}\right]^{-1}, \tag{34}$$

and

$$\ln\left[\frac{(4S_1+\beta_{\rm c}\omega_1)(4S_2+\beta_{\rm h}\omega_2)}{(4S_2+\beta_{\rm c}\omega_1)(4S_1+\beta_{\rm h}\omega_2)}\right] - \frac{4(S_2-S_1)\beta_{\rm c}\omega_1}{(4S_1+\beta_{\rm c}\omega_1)(4S_2+\beta_{\rm c}\omega_1)} = 0. \tag{35}$$

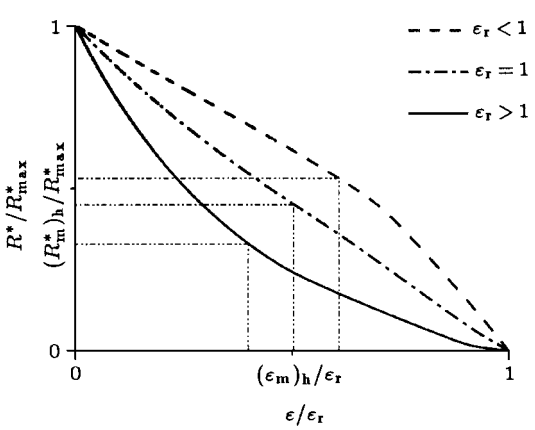


Fig.6. The dimensionless cooling rate R^* versus the coefficient of performance ε in the high-temperature limit. Dashed ($\omega_{\rm h}$=6.0J), dash-dotted ($\omega_{\rm h}$=4.0J) and solid ($\omega_{\rm h}$=2.0J) curves are presented for the parameters $kT_{\rm h}$=200J, $kT_{\rm c}$=150J, $\omega_{\rm c}$=2.0J, and $q = -0.5$. The parameters $(R_{\rm m})_{\rm h}$ and $(\varepsilon_{\rm m})_{\rm h}$ represent the cooling rate and coefficient of performance at the maximum ($R\varepsilon$), respectively.

Using Eqs.(32), (33) and (35), one can generate some fundamental optimal relation curves R^*–ε for given β_h, β_c, ω_h and ω_c, as shown in Fig.6, where the parameters kT_h=200J, kT_c=150J, ω_h=6.0J (dashed curve), ω_h=4.0J (dash-dotted curve), ω_h=3.0J (solid curve) and ω_c=2.0J are adopted. It is clearly seen from the curves in Fig.6 that at high temperatures, the R^*–ε curves of an irreversible spin quantum Brayton refrigeration cycle are different from those of the refrigeration cycle at low temperatures. The cooling rate R decreases monotonically as the coefficient of performance increases. When $\varepsilon_r \approx 1$, the R^*-ε curve is approximately a straight line, i.e. the cooling rate is approximately a linearly decreasing function of the coefficient of performance. When the condition $\varepsilon_r \approx 1$ is not satisfied, the R^*-ε curve is convex for $\varepsilon_r > 1$, while the R^*-ε curve is concave for $\varepsilon_r < 1$. When $\varepsilon \to 0$, the cooling rate attains its maximum value R_{max}. The maximum cooling rate

$$R_{max} = \frac{4a(S_2 - S_1)\omega_{1m}}{\ln[(4S_1 + \beta_c\omega_{1m})/(4S_2 + \beta_c\omega_{1m})]} \tag{36}$$

can be calculated from Eqs.(33) and (35), where

$$\omega_{1m} = (4S_1 + \beta_c\omega_{1m})(4S_2 + \beta_c\omega_{1m}) \times \ln\frac{[(4S_1 + \beta_c\omega_{1m})/(4S_2 + \beta_c\omega_{1m})]}{[4(S_2 - S_1)\beta_c]}.$$

In such a case, an infinite power input is required. This indicates that the cooling rate of a practical refrigerator cannot approach R_{max}. Thus, the states of $R = R_{max}$ and $\varepsilon = \varepsilon_r$ are two limited states and the refrigeration cycle cannot be operated in the two states, so that consideration must be given to both the coefficient of performance and the cooling rate. Equations (32) and (33) just provide some theoretical bases for the question of how to choose reasonably the two parameters. For example, when we pay equal attention to both the coefficient of performance ε and the cooling rate R, the product εR may be taken as an objective function.[19] The cooling rate $(R_m)_h$ and coefficient of performance $(\varepsilon_m)_h$ under the maximum εR condition can be calculated, as shown in Fig.6. It is clearly seen from the curves in Fig.6 that there exists a relation: $(\varepsilon_m)_h/\varepsilon_r \approx (R_m)_h/R_{max} \approx 1/2$ for $\varepsilon_r = 1$.

5. Discussion

1. It is clearly seen from Eqs.(9), (10), (14), (19), and (20) that the cycle period is a monotonically decreasing function of the parameters a and q, while the cooling rate R and power input P are monotonically increasing functions of the parameters a and q. This implies that the larger the parameters a and q are, the larger the cooling rate R and power input P will be. However, the two dimensionless parameters R^* and P^* are independent of the parameter a. Thus, Figs.3, 4 and 6 are true for an arbitrary value of the parameter a.

2. When the working substance is composed of a spin-J system (J=1/2, 1, 3/2, 2, $\cdots$), the mean value of the spin angular momentum can be written as[20,21]

$$S = \langle \hat{S}_z \rangle = -JB_J(\beta'\omega J), \tag{37}$$

where $-J \le S \le J$ and $B_J(x) = [(2J+1)/(2J)]\coth[(2J+1)x/(2J)] - (1/2J)\coth[x/(2J)]$ is the Brillouin function. Based on the quantum master equation and semi-group approach, one can prove that the general expression of time evolution for a spin-J system coupling with the heat reservoir and the external magnetic field is given by

$$t = -\frac{1}{a}\int_{S_i}^{S_f} \frac{dS}{e^{q\beta\omega}\{2(1+e^{\beta\omega})S + 2[J(J+1) - M](e^{\beta\omega} - 1)\}}, \tag{38}$$

where $M = \langle \hat{S}_z^2 \rangle$ and $\langle \hat{S}^2 \rangle = J(J+1)$. Using a similar method to that mentioned above and Eqs.(38) and (19)–(21), we can analyse the performance characteristics of an irreversible quantum refrigeration cycle using spin-J systems as the working substance. At high temperatures, Eqs.(37) and (38) may be, respectively, simplified as

$$S = -\frac{J(J+1)}{3}\beta'\omega \tag{39}$$

and

$$t = -\frac{1}{2a}\int_{S_i}^{S_f} \frac{dS}{2S + \beta\omega[J(J+1) - M]}, \tag{40}$$

where $M = J(J+1)/3$. Comparing Eq.(39) with Eq.(30), we can find that the heat amounts of the two isomagnetic field processes in the cycle may be obtained by multiplying the factor of $4J(J+1)/3$ in Eqs.(15) and (16). On the other hand, it can be found form Eqs.(39) and (40) that the cycle period is still given by Eq.(31). Thus, at high temperatures, the co-

efficient of performance of the cycle using the spin-J systems as the working substance is the same as that of the cycle working with the spin-1/2 systems, while the cooling rate and power input are $4J(J+1)/3$ times those of the quantum cycle working with the spin-1/2 systems.

3. The above discussion only refers to a single spin-J system. For the working substance consisting of many noninteracting spin-J systems, the coefficient of performance is still true, while the work input, power input, and amounts of heat exchange can be obtained as long as the above results are simply multiplied by the total number of spin systems.

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