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New localized excitations in a (2+1)-dimensional Broer–Kaup system*

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Starting with the extended homogeneous balance method and a variable separation approach, a general variable separation solution of the Broer–Kaup system is derived. In addition to the usual localized coherent soliton excitations like dromions, lumps, rings, breathers, instantons, oscillating soliton excitations, peakon and fractal localized solutions, some new types of localized excitations, such as compacton and folded excitations, are obtained by introducing appropriate lower-dimensional piecewise smooth functions and multiple-valued functions, and some interesting novel features of these structures are revealed.

Keywords: extended homogeneous balance method, variable separation approach, localized excitations, (2+1)-dimensional, Broer–Kaup system

PACC: 0545, 0230, 0340, 0365

1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of sciences such as biology, chemistry, mathematics, communication and especially in almost all branches of physics like condensed matter physics, field theory, fluid dynamics, plasma physics and optics, etc.[1−3] Searching for exact and explicit solutions of a nonlinear physical model, especially for new exponentially localized structures like soliton solutions or for the excitations with novel properties is a very significant work. Now several significant (2+1)-dimensional models[4−12] have been investigated and some special types of localized solutions for these models have also been obtained by means of different approaches: for example, the bilinear method, standard and extended truncated Painlevé analysis, variable separation approach, standard and extended homogeneous balance method, and so on.[13−20] From the above studies of (2+1)-dimensional models, one can find that there exist more abundant localized structures than in lower dimensions. This fact implies that there may exist similar or new localized structures that are unrevealed in other (2+1)-dimensional integrable models.

In this paper, we use the extended homogeneous balance method (EHBM) and variable separation approach (VSA) to study the (2+1)-dimensional Broer–Kaup (BK) system,

$$H_{ty} = H_{xxy} - 2(HH_x)_y - 2G_{xx}, \tag{1a}$$

$$G_t = -G_{xx} - 2(GH)_x, \tag{1b}$$

which are obtained from the Kadomtsev–Petviashvili (KP) equation with symmetry constraints,[21] and a general variable separation solution of the BK system is derived. Some types of the usual localized excitations of Eqs.(1) such as dromions, lumps, ring soliton and oscillated dromion, breathers solution, fractal-dromion and fractal-lump soliton structures,[12] etc, can be easily constructed by selecting appropriate arbitrary functions. In addition to the usual localized structures, some new localized excitations like compactons, foldons are found by selecting some types of lower-dimensional appropriate functions.

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2. Variable separation approach based on extended homogeneous balance method for the (2+1)-dimensional BK system

According to the EHBM, let

$$H = f'(w)w_x + a(x,t), \tag{2a}$$
$$G = g''(w)w_x w_y + g'(w)w_{xy}, \tag{2b}$$

where the prime denotes d/dw and $f(w), g(w)$ and $a(x,t)$ are undetermined functions. Substituting Eqs.(2) into Eqs.(1), we obtain

$$H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = (2f''^2 + 2f'f^{(3)} - f^{(4)} + 2g^{(4)})w_y w_x^3 + \text{lower power terms of the derivatives of } w(x,y,t) \text{ with respect to } x,\ y \text{ and } t{=}0. \tag{3a}$$

$$G_t + G_{xx} + 2(GH)_x = (g^{(4)} + 2f'g^{(3)} + 2g''f'')\, w_y w_x^3 + \text{lower power terms of the derivatives of } w(x,y,t) \text{ with respect to } x,\ y \text{ and } t{=}0. \tag{3b}$$

Setting the coefficients of $w_y w_x^3$ in Eqs.(3) to zero yields an ordinary differential equation system

$$2f''^2 + 2f'f^{(3)} - f^{(4)} + 2g^{(4)} = 0, \tag{4a}$$
$$g^{(4)} + 2f'g^{(3)} + 2g''f'' = 0. \tag{4b}$$

The following special solutions exist for Eqs.(4):

$$f(w) = g(w) = \ln(w). \tag{5}$$

Thereby we have $f'f'' = -\frac{1}{2}f^{(3)}$, $f'^2 = -f^{(2)}$. Using these results, expression (3) can be simplified to

$$\begin{aligned} &H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} \\ =&f'(w_t + w_{xx} + 2aw_x)_{xy} \\ &+ f''[w_{xy}(w_t + w_{xx} + 2aw_x) \\ &+ w_x(w_t + w_{xx} + 2aw_x)_y \\ &+ w_y(w_t + w_{xx} + 2aw_x)_x] \\ &+ f^{(3)}[w_x w_y(w_t + w_{xx} + 2aw_x)] = 0, \end{aligned} \tag{6a}$$

$$\begin{aligned} &G_t + G_{xx} + 2(GH)_x \\ =&g'(w_t + w_{xx} + 2aw_x)_{xy} \\ &+ g''[w_{xy}(w_t + w_{xx} + 2aw_x) \\ &+ w_x(w_t + w_{xx} + 2aw_x)_y \\ &+ w_y(w_t + w_{xx} + 2aw_x)_x] \\ &+ g^{(3)}[w_x w_y(w_t + w_{xx} + 2aw_x)] = 0, \end{aligned} \tag{6b}$$

Setting the coefficients of $f^{(3)}$, f'', f' and $g^{(3)}$, g'', g' in Eqs.(6) to zero and simplifying then yield a set of partial differential equations for $w(x,y,t)$:

$$(w_t + w_{xx} + 2aw_x)_{xy} = 0, \tag{7a}$$
$$w_x w_y(w_t + w_{xx} + 2aw_x) = 0, \tag{7b}$$
$$\begin{aligned} &w_{xy}(w_t + w_{xx} + 2aw_x) \\ &+ w_x(w_t + w_{xx} + 2aw_x)_y \\ &+ w_y(w_t + w_{xx} + 2aw_x)_x = 0. \end{aligned} \tag{7c}$$

Analysing Eqs.(7), we find that when $w_x \neq 0$ and $w_y \neq 0$, the independent equation in Eqs.(7) reads

$$w_t + w_{xx} + 2aw_x = 0. \tag{8}$$

To find some exact solution of Eq.(8) via the variable separation approach, we look for the solution of Eq.(8) in the form

$$w(x,y,t) = 1 + c_1 p(x,t) + c_2 q(y) + C p(x,t) q(y), \tag{9a}$$

where $p(x,t)$ is an arbitrary function of variables $\{x, t\}$, $q(y)$ is an arbitrary function of variables $\{y\}$, and c_1, c_2 and C are three arbitrary constants. It is clear that the variables x and y now have been separated totally.

Substituting ansatz (9a) into Eq.(8), we have

$$c_1(p_t + p_{xx} + 2ap_x) + Cq(p_t + p_{xx} + 2ap_x) = 0. \tag{9b}$$

To solve Eq.(9b) is quite easy because of the arbitrariness of the function $a(x,t)$. In other words, if we fix the function $a(x,t)$ as

$$a(x,t) = -\frac{p_t + p_{xx}}{2p_x}, \tag{9c}$$

then p and q become two arbitrary functions.

Finally, substituting Eq.(5) into Eq.(2) along with Eqs.(9a) and (9c), we obtain a quite general excitation of the (2+1)-dimensional BK system

$$H(x,y,t) = \frac{(c_1 + Cq)p_x}{1 + c_1 p + c_2 q + Cpq} - \frac{p_t + p_{xx}}{2p_x}, \tag{10}$$
$$G(x,y,t) = \frac{(A - c_1 c_2)p_x q_y}{(1 + c_1 p + c_2 q + Cpq)^2}, \tag{11}$$

with two arbitrary functions $p(x,t)$ and $q(y)$.

In Ref.[12], it was pointed out that the BK system possesses some special types of coherent structures for potential quantity G rather than the physical field H. Moreover, as the arbitrariness of the functions p and q are included in Eq.(11), G obviously possesses quite rich structures such as multi-dromion solutions, multi-lump solutions, multi-ring soliton solutions, breathers, instantons, oscillating dromion solutions, peakon and fractal localized solutions. Now the important matter

is whether we can find some new types of localized excitations for potential quantity G.

3. Some new localized excitations of the (2+1)-dimensional BK system

3.1. Compacton solutions and their interaction behaviour

It is well known that, in addition to the continuous localized excitations in (1+1)-dimensional nonlinear systems, some types of significantly weak solutions such as the compacton solutions have attracted much attention of both mathematicians and physicists. The so-called (1+1)-dimensional compacton solutions, which describe the typical (1+1)-dimensional soliton solutions with a finite wavelength when the nonlinear dispersion effects are included, were first given by Rosenau and Hyman,[22] while a few papers have investigated higher-dimensional compactons.[23–25] Because of the introduction of arbitrary functions in expression (11), we can find some types of multiple compacton solutions by selecting the arbitrary functions appropriately. For instance, if we fix the functions p and q as

$$
\begin{aligned}
&p = 0, \quad x + v_i t \le x_{0i} - \frac{\pi}{2k_i},\\
&p = \sum_{i=1}^{M} a_i \sin\left(k_i(x + v_i t - x_{0i})\right) + a_i, \quad x_{0i} - \frac{\pi}{2k_i} < x + v_i t \le x_{0i} + \frac{\pi}{2k_i},\\
&p = \sum_{i=1}^{M} 2a_i, \quad x + v_i t > x_{0i} + \frac{\pi}{2k_i};
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&q = 0, \quad y \le y_{0j} - \frac{\pi}{2l_j},\\
&q = \sum_{j=1}^{N} b_j \sin\left(l_j(y - y_{0j})\right) + b_j, \quad y_{0j} - \frac{\pi}{2l_j} < y \le y_{0j} + \frac{\pi}{2l_j},\\
&q = \sum_{j=1}^{N} 2b_j, \quad y > y_{0j} + \frac{\pi}{2l_j},
\end{aligned}
\tag{13}
$$

where a_i, k_i, v_i, b_j, l_j, x_{0i} and y_{0j} are all arbitrary constants, then solution (11) together with Eqs.(12) and (13) becomes a multi-compacton solution.

Selecting M=2, N=2, $A = -1/200$, $c_1 = c_2 = 1/20$, $C = 1/500$, $a_1 = -1.2$, $a_2 = -1$, $b_1 = b_2 = 1$, $k_1 = k_2 = 1$, $v_1 = -1$, $v_2 = 3$, $l_1 = l_2 = 1$, $x_{01} = x_{02} = 0$, $y_{01} = 0$, $y_{02} = 5$, we can obtain a four-compacton structure for the (2+1)-dimensional BK system. Figures 1(a)–1(f) show the evolution behaviour of interaction among the four compactons. We see that the interaction among the four compactons exhibits a new phenomenon: that is, the interaction among them is non-elastic but they do not completely exchange their shapes after the interaction.

3.2. Folded solitary waves, foldon solutions and their interaction behaviour

Because the real natural phenomena are very complicated, in some cases, it is even impossible to describe the natural phenomena by single-valued functions. For instance, in the real natural world, there exist very complicated folded phenomena such as the folded protein, folded brain and skin surface, and many other kinds of folded biological systems. The simplest foldons are the so-called loop solitons,[26,27] which can be found in many (1+1)-dimensional integrable systems[26,27] and have been employed in some possible physical fields like the string interaction with external field,[28] quantum field theory,[29,30] and particle physics.[31,32] However, the problem of how to find some folded localized excitations and/or foldons in higher-dimensional physical models is still open. Now we discuss the folded localized excitation in a (2+1)-dimensional BK system.

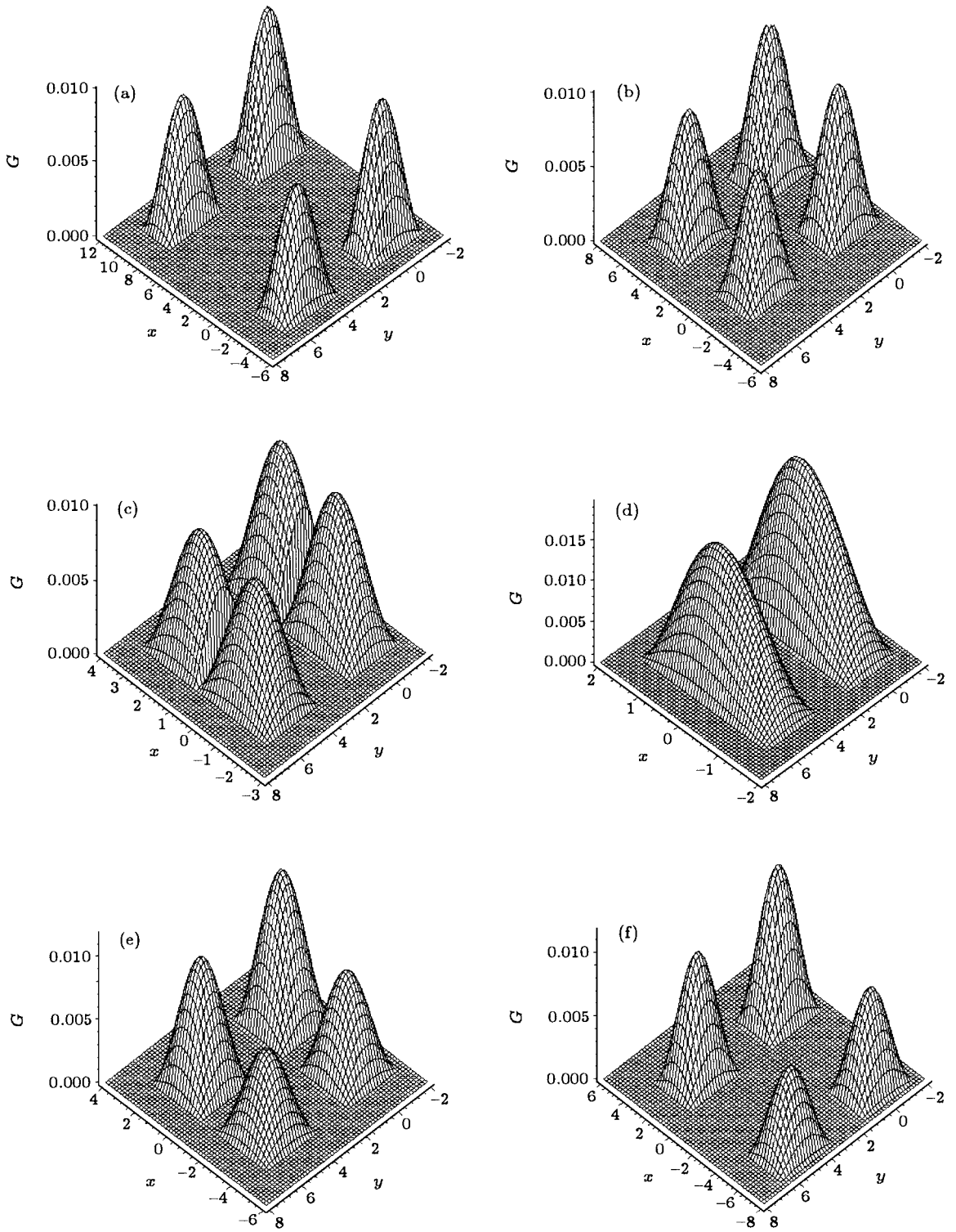


Fig.1. Evolution of a four-compacton solution determined by Eq.(11) at (a) $t = -3$ (b) $t = -1.5$, (c) $t = -0.6$,(d) $t = 0$, (e) $t = 1$, and (f) $t = 2$.

In order to construct some kinds of interesting folded localized excitations and/or foldons for the potential quantity G, we should introduce some suitable multi-valued functions. For example,

$$p_x = \sum_{j=1}^{M} U_j(\xi + w_j t),$$
$$x = \xi + \sum_{j=1}^{M} X_j(\xi + w_j t), \tag{14}$$

where U_j and X_j are localized excitations with properties $U_j(\pm\infty) = 0$, $X_j(\pm\infty) =$const. From Eq.(14), one can find that ξ may be a multi-valued function in some suitable regions of x by selecting the functions X_j appropriately. Therefore, the function p_x, which is obviously an interaction solution of M localized excitations since we have the property $\xi\,|_{x\to\infty} \to \infty$, may be a multi-valued function of x in these areas though it is a single-valued function of ξ. Actually, most of the known multi-loop solutions are special cases of Eq.(14). Similarly, we also treat the function $q(y)$ in the following way,

$$q_y = \sum_{j=1}^{N} V_j \eta,$$
$$y = \eta + \sum_{j=1}^{N} Y_j \eta. \tag{15}$$

In Fig.2, five typical folded solitary waves are plotted for the potential quantity G determined by Eq.(11) with the functions selected as

$$\begin{aligned}
&p_x = -10\mathrm{sech}^2(\xi + wt),\\
&p = -\frac{7\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 9,\\
&x = \xi - 1.15\tanh(\xi + wt),\\
&q_y = -\mathrm{sech}^2(\eta),\\
&q = -\frac{5\sinh(\eta)}{3\cosh(\eta)} - \frac{\sinh(\eta)}{3\cosh^3(\eta)},\\
&y = \eta + \tanh(\eta);
\end{aligned} \tag{16}$$

$$\begin{aligned}
&p_x = -\mathrm{sech}^2(\xi + wt),\\
&p = \frac{2\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{5\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 0.9,\\
&x = \xi - 2.5\tanh(\xi + wt),\\
&q_y = -\mathrm{sech}^2(\eta),\\
&q = -\frac{\sinh(\eta)}{\cosh(\eta)},\\
&y = \eta;
\end{aligned} \tag{17}$$

$$\begin{aligned}
&p_x = -\mathrm{sech}^2(\xi + wt),\\
&p = -\frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{7\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 0.9,\\
&x = \xi - 1.4\tanh(\xi + wt),\\
&q_y = -\mathrm{sech}^2(\eta),\\
&q = \frac{2\sinh(\eta)}{3\cosh(\eta)} + \frac{5\sinh(\eta)}{6\cosh^3(\eta)},\\
&y = \eta - 2.5\tanh(\eta);
\end{aligned} \tag{18}$$

$$\begin{aligned}
&p_x = -\mathrm{sech}^2(\xi + wt),\\
&p = \frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{8\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 7,\\
&x = \xi - 1.6\tanh(\xi + wt),\\
&q_y = -\mathrm{sech}^2(\eta),\\
&q = \frac{\sinh(\eta)}{15\cosh(\eta)} + \frac{8\sinh(\eta)}{15\cosh^3(\eta)},\\
&y = \eta - 1.6\tanh(\eta);
\end{aligned} \tag{19}$$

$$\begin{aligned}
&p_x = -\mathrm{sech}^2(\xi + wt),\\
&p = -\frac{7\sinh(\xi + wt)}{30\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{60\cosh^3(\xi + wt)} + 3,\\
&x = \xi - 1.15\tanh(\xi + wt),\\
&q_y = -\mathrm{sech}^2(\eta),\\
&q = -\frac{7\sinh(\eta)}{30\cosh(\eta)} + \frac{23\sinh(\eta)}{60\cosh^3(\eta)},\\
&y = \eta - 1.15\tanh(\eta).
\end{aligned} \tag{20}$$

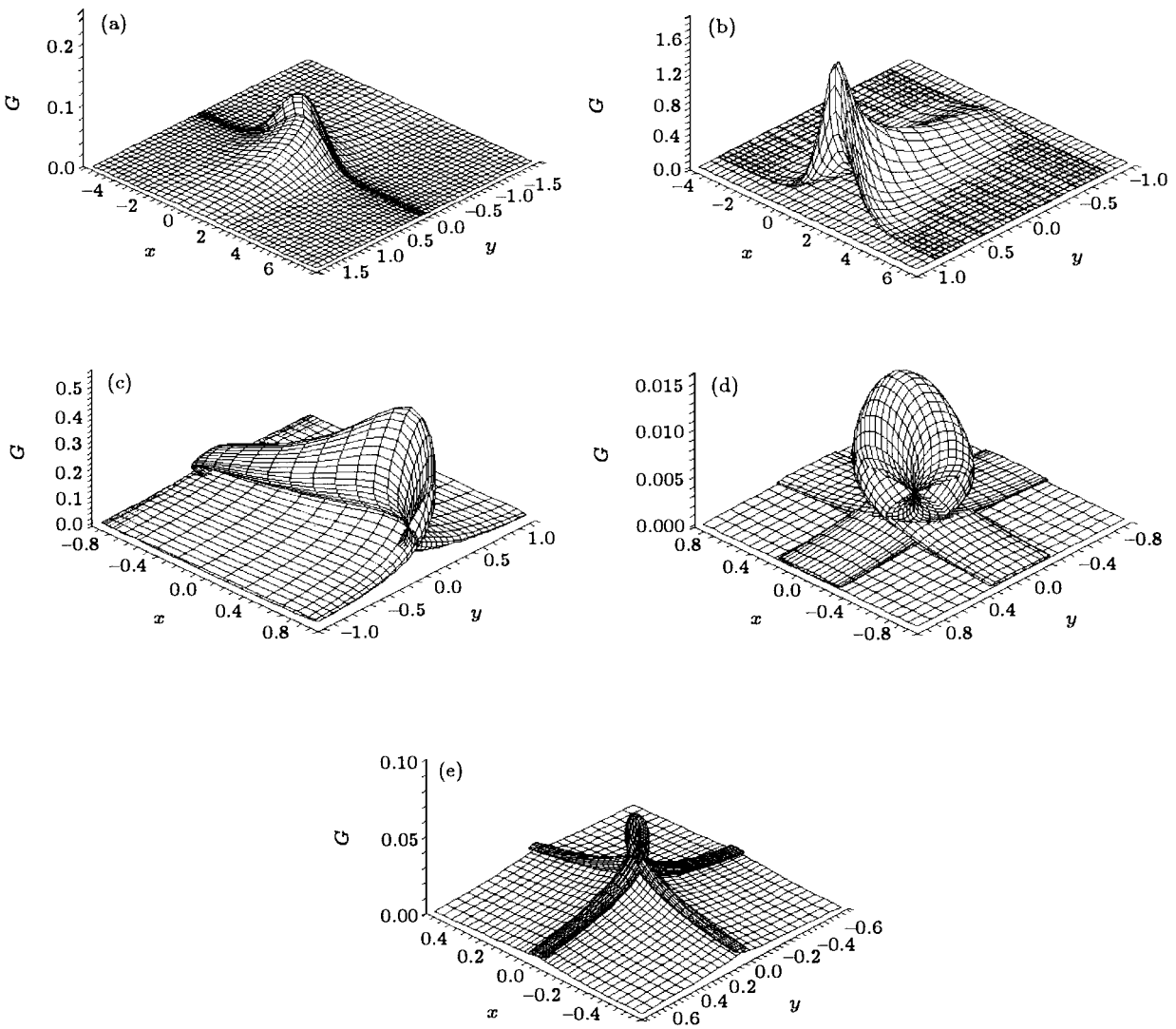


Fig.2. Five typical folded solitary waves for the potential quantity G determined by Eq.(11) at $t = 0$ together with Eqs.(16)–(20) for $A = 2, a_1 = a_2 = 1, a_3 = 1/25$, as shown in (a), (b), (c), (d), and (e).

Figure 3 shows an evolution plot of the two foldons for the potential quantity G determined by Eq.(11) with the functions selected as

$$p_x = -\frac{4}{5}\mathrm{sech}^2(\xi) - \frac{1}{2}\mathrm{sech}^2(\xi - t),$$
$$x = \xi - 1.5\tanh(\xi) - 1.5\tanh(\xi - t),$$
$$q_y = -\mathrm{sech}^2(\eta),$$
$$y = \eta - 2\tanh(\eta). \tag{21}$$

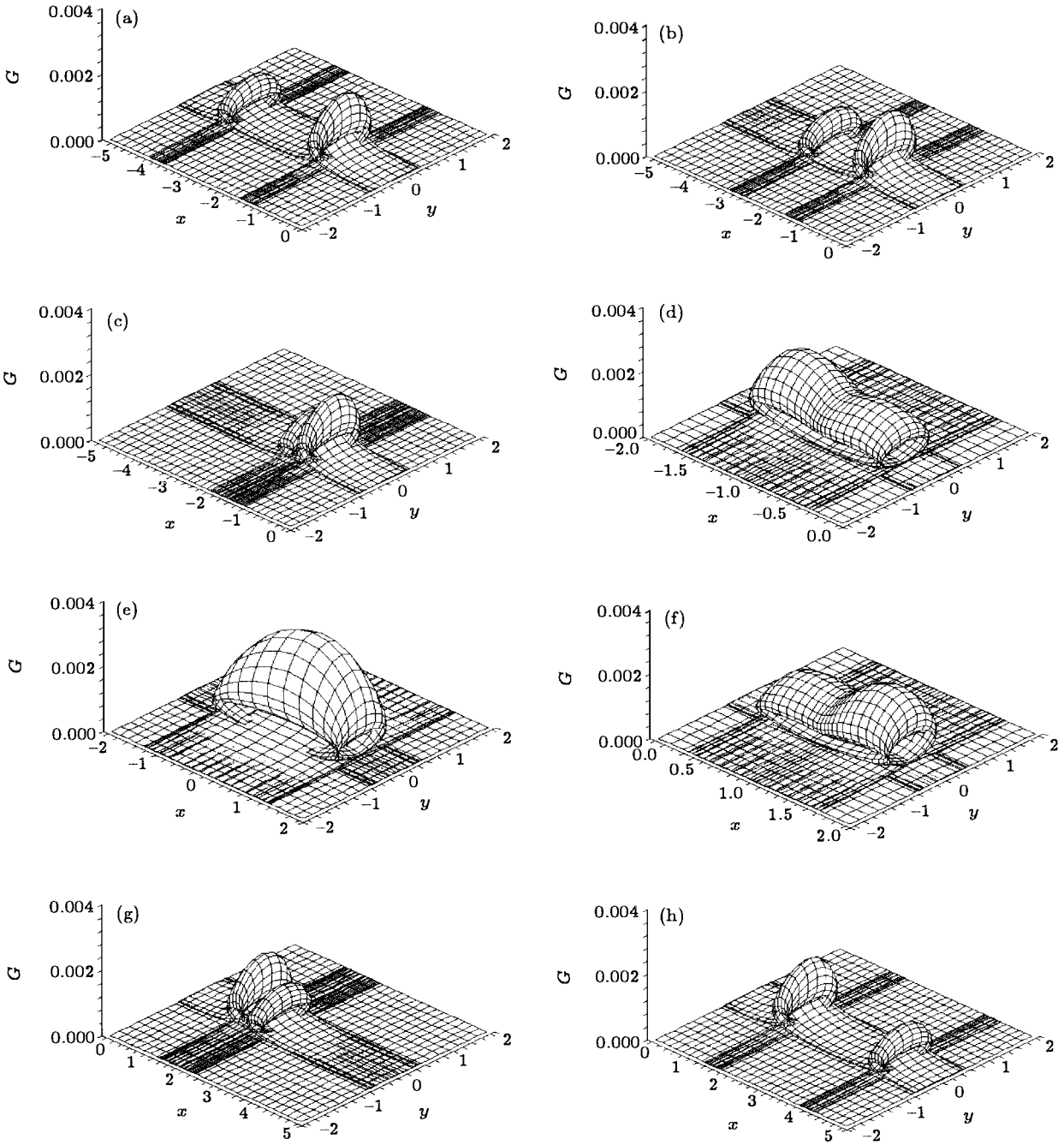


Fig.3. Evolution plots of two foldons for the potential quantity G determined by Eq.(11) together with functions (21) at time (a) $t = -5.5$, (b) $t = -4.5$, (c) $t = -3.5$, (d) $t = -2$, (e) $t = 0$, (f) $t = 2$, (g) $t = 3.5$, and (h) $t = 5.5$.

4. Summary

In summary, with the help of the extended homogeneous balance method and variable separation approach, the (2+1)-dimensional BK system has been solved. Abundant localized coherent soliton structures of the solution (11), such as multi-dromion, multi-ring, multi-lump solutions, breathers, instantons, chaotic and fractal soliton solutions, etc, can be easily constructed by selecting appropriate arbitrary functions. In addition to the usual localized coherent soliton excitations, some new types of localized excitations, such as compacton and folded excitations, are obtained by introducing appropriate lower-dimensional piecewise smooth functions and multiple-valued functions, and some interesting novel features of these structures are revealed. This work is only a first step; further study to find the localized excitations like new types of folded solitary waves, foldons and their practical applications is still necessary.

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