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Separability criteria for multipartite quantum mixed states*

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The separability of mixed states in multipartite quantum systems is investigated. If a quantum state in a multipartite system with an arbitrary dimension is separable, some quantity in relation to Hermitian matrix is positive.

Keywords: separability, density matrix, multipartite mixed state
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1. Introduction

Quantum entanglement has played a very important role in quantum information processing such as quantum teleportation,[1] quantum cryptography,[2] quantum dense coding[3] and parallel computing.[4] However, the physical characteristics and mathematical structures of entangled states have not been well understood.

For pure states the separability is quite well understood. However, for multipartite quantum mixed states even no well defined measure of entanglement exists, for the separability problem is more complicated. A multipartite quantum state $\rho_{A_1A_2\ldots A_n}$ is called separable (not entangled) if it can be written as

$$\rho_{A_1A_2\ldots A_n} = \sum_i q_i {\rho_i}^{A_1} \otimes {\rho_i}^{A_2} \otimes \cdots \otimes {\rho_i}^{A_n}, \quad (1)$$

where ${\rho_i}^{A_1}, {\rho_i}^{A_2}, \ldots, {\rho_i}^{A_n}$ are the density matrices for systems $A_1, A_2, \ldots, A_n$ respectively, $q_i > 0$ and $\sum_i q_i = 1$. This is equivalent to the condition

$$\rho_{A_1A_2\ldots A_n} = \sum_i p_i |\psi_i^{A_1}\rangle\langle\psi_i^{A_2}| \otimes |\phi_i^{A_2}\rangle\langle\phi_i^{A_2}| \otimes \cdots \otimes |\mu_i^{A_n}\rangle\langle\mu_i^{A_n}|, \quad (2)$$

where $|\psi_i^{A_1}\rangle, |\phi_i^{A_2}\rangle, \ldots, |\mu_i^{A_n}\rangle$ are normalized pure states of systems $A_1, A_2, \ldots, A_n$ respectively, $p_i > 0$ and $\sum_i p_i = 1$. Much effort has been made recently to analyse the separability and quantitative characteristics of quantum entanglement. Werner[5] has pointed out first that separable states must satisfy all possible Bell inequalities. Peres[6] has discovered a very strong necessary condition for separability (the PPT criterion). This criterion is further shown to be also sufficient for 2×2 and 2×3 bipartite systems.[7,8] In view of the quantitative characteristics for entanglement, Bennett, *et al* have succeeded in computing the "entanglement of formation"[9] and thus Wootters has obtained a separability criterion for 2×2 mixed states.[10] The "reduction criterion"[11,12] is equivalent to the PPT criterion for $2 \times N$ composite systems but is not sufficient for separability in general cases. Besides, there are also many other criteria such as majorization,[13] entanglement witnesses,[7,14,15] and matrix realignment.[16,17]

The separability in $2 \times N$ quantum systems[18] and the separability in $R \times N$ (or $2 \times 2 \times N$) quantum systems[19] have been investigated in terms of Hermitian matrix. In this paper we generalize the results. In Section 2, we show that some quantity in relation to Hermitian matrix is positive if a quantum state in $2\times2\times\cdots\times2\times N$ quantum systems is separable. In Section 3, we generalize the results to $R\times R\times\cdots\times R\times N$ quantum systems.

2. Separability in $2\times2\times\cdots\times2\times N$ quantum systems

We first consider the separability of the mixed states in multipartite quantum systems with the dimension $2 \times 2 \times 2 \times N$.

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Let the dimensions of systems A_1, A_2, A_3 and A_4 be 2, 2, 2 and N, respectively. A given state $\rho_{A_1A_2A_3A_4}$ can be written as

$$\begin{aligned}\rho_{A_1A_2A_3A_4} =& I_2\otimes I_2\otimes I_2\otimes L_0\\ &+\sum_{j_1=1}^{3}\lambda_{j_1}^{(1)}\otimes I_2\otimes I_2\otimes L_{j_1}^{(1)}\\ &+\sum_{j_2=1}^{3} I_2\otimes\lambda_{j_2}^{(2)}\otimes I_2\otimes L_{j_2}^{(2)}\\ &+\sum_{j_3=1}^{3} I_2\otimes I_2\otimes\lambda_{j_3}^{(3)}\otimes L_{j_3}^{(3)}\\ &+\sum_{j_1,j_2=1}^{3}\lambda_{j_1}^{(1)}\otimes\lambda_{j_2}{}^{(2)}\otimes I_2\otimes L_{j_1j_2}^{(12)}\\ &+\sum_{j_1,j_3=1}^{3}\lambda_{j_1}^{(1)}\otimes I_2\otimes\lambda_{j_3}^{(3)}\otimes L_{j_1j_3}^{(13)}\\ &+\sum_{j_2,j_3=1}^{3} I_2\otimes\lambda_{j_2}^{(2)}\otimes\lambda_{j_3}^{(3)}\otimes L_{j_2j_3}^{(23)}\\ &+\sum_{j_1,j_2,j_3=1}^{3}\lambda_{j_1}^{(1)}\otimes\lambda_{j_2}^{(2)}\otimes\lambda_{j_3}^{(3)}\otimes L_{j_1j_2j_3}^{(123)},\end{aligned}\tag{3}$$

where $\lambda_{j_r}^{(r)}$ $(j_r=1,2,3; r=1,2,3)$ are Pauli matrices, L_0, $L_{j_1}^{(1)}$, $L_{j_2}^{(2)}$, $L_{j_3}^{(3)}$, $L_{j_1j_2}^{(12)}$, $L_{j_1j_3}^{(13)}$, $L_{j_2j_3}^{(23)}$, $L_{j_1j_2j_3}^{(123)}$, $(j_1,j_2,j_3=1,2,3)$ are N-dimensional Hermitian matrices. A separable state

$$\begin{aligned}\rho_{A_1A_2A_3A_4} =& \sum_i p_i|\psi_i^{A_1}\rangle\langle\psi_i^{A_1}|\otimes|\phi_i^{A_2}\rangle\langle\phi_i^{A_2}|\\ &\otimes|v_i^{A_3}\rangle\langle v_i^{A_3}|\otimes|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\end{aligned}\tag{4}$$

can be also written as

$$\begin{aligned}\rho_{A_1A_2A_3A_4} =& \sum_i p_i\frac{1}{2}\left(I_2+\sum_{j_1=1}^{3}a_{ij_1}^{(1)}\lambda_{j_1}^{(1)}\right)\\ &\otimes\frac{1}{2}\left(I_2+\sum_{j_2=1}^{3}a_{ij_2}^{(2)}\lambda_{j_2}^{(2)}\right)\\ &\otimes\frac{1}{2}\left(I_2+\sum_{j_3=1}^{3}a_{ij_3}^{(3)}\lambda_{j_3}^{(3)}\right)\otimes|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\end{aligned}\tag{5}$$

where $(a_{i1}^{(1)}, a_{i2}^{(1)}, a_{i3}^{(1)})$, $(a_{i1}^{(2)}, a_{i2}^{(2)}, a_{i3}^{(2)})$ and $(a_{i1}^{(3)}, a_{i2}^{(3)}, a_{i3}^{(3)})$ are real vectors on the Bloch sphere and satisfy

$$\sum_{j_1=1}^{3}\left(a_{ij_1}^{(1)}\right)^2=1,\quad \sum_{j_2=1}^{3}\left(a_{ij_2}^{(2)}\right)^2=1,\quad \sum_{j_3=1}^{3}\left(a_{ij_3}^{(3)}\right)^2=1.\tag{6}$$

Comparing Eqs. (3) and (5), we have

$$\begin{aligned}L_0 &= \frac{1}{8}\sum_i p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_1}^{(1)} &= \frac{1}{8}\sum_i a_{ij_1}^{(1)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_2}^{(2)} &= \frac{1}{8}\sum_i a_{ij_2}^{(2)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_3}^{(3)} &= \frac{1}{8}\sum_i a_{ij_3}^{(3)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_1j_2}^{(12)} &= \frac{1}{8}\sum_i a_{ij_1}^{(1)}a_{ij_2}^{(2)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_1j_3}^{(13)} &= \frac{1}{8}\sum_i a_{ij_1}^{(1)}a_{ij_3}^{(3)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_2j_3}^{(23)} &= \frac{1}{8}\sum_i a_{ij_2}^{(2)}a_{ij_3}^{(3)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|,\\ L_{j_1j_2j_3}^{(123)} &= \frac{1}{8}\sum_i a_{ij_1}^{(1)}a_{ij_2}^{(2)}a_{ij_3}^{(3)}p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|.\end{aligned}\tag{7}$$

Let $\sum_{j=1}^{63} l_jL_j$ and vector $(l_1,l_2,\ldots,l_{63})$ denote

$$\begin{aligned}&\sum_{r=1}^{3}\sum_{j_r=1}^{3}l_{j_r}^{(r)}L_{j_r}^{(r)}+\sum_{j_1,j_2=1}^{3}l_{j_1j_2}^{(12)}L_{j_1j_2}^{(12)}\\ &+\sum_{j_1,j_3=1}^{3}l_{j_1j_3}^{(13)}L_{j_1j_3}^{(13)}+\sum_{j_2,j_3=1}^{3}l_{j_2j_3}^{(23)}L_{j_2j_3}^{(23)}\\ &+\sum_{j_1,j_2,j_3=1}^{3}l_{j_1j_2j_3}^{(123)}L_{j_1j_2j_3}^{(123)}\end{aligned}$$

and $(l_1^{(1)},l_2^{(1)},\ldots,l_{333}^{(123)})$ respectively, we will be able to deduce the following theorems.

Theorem 1 If $\rho_{A_1A_2A_3A_4}$ is separable, then $4L_0-\sum_{j=1}^{63}l_jL_j$ is positive for any vector $(l_1,l_2,\ldots,l_{63})$ of unit length or less.

Proof From Eqs. (7), we can acquire

$$\begin{aligned}&4L_0-\sum_{j=1}^{63}l_jL_j\\ =&\frac{1}{8}\sum_i p_i(4-\sum_{r=1}^{3}\sum_{j_r=1}^{3}l_{j_r}^{(r)}a_{ij_r}^{(r)}-\sum_{j_1,j_2=1}^{3}l_{j_1j_2}^{(12)}a_{ij_1}^{(1)}a_{ij_2}^{(2)}\\ &-\sum_{j_1,j_3=1}^{3}l_{j_1j_3}^{(13)}a_{ij_1}^{(1)}a_{ij_3}^{(3)}-\sum_{j_2,j_3=1}^{3}l_{j_2j_3}^{(23)}a_{ij_2}^{(2)}a_{ij_3}^{(3)}\\ &-\sum_{j_1,j_2,j_3=1}^{3}l_{j_1j_2j_3}^{(123)}a_{ij_1}^{(1)}a_{ij_2}^{(2)}a_{ij_3}^{(3)})|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|.\end{aligned}\tag{8}$$

Applying Eqs.(6) and the assumption of the theorem,

we can obtain

$$\begin{aligned}\sum_{j=1}^{63} l_j L_j \le & \frac{1}{8}\sum_i p_i \frac{1}{2}\Bigg[\sum_{j=1}^{63}(d_j)^2+\sum_{r=1}^{3}\sum_{j_r=1}^{3}\left(a_{ij_r}^{(r)}\right)^2 \\ & +\sum_{j_1,j_2=1}^{3}\left(a_{ij_1}^{(1)}a_{ij_2}^{(2)}\right)^2 \\ & +\sum_{j_1,j_3=1}^{3}\left(a_{ij_1}^{(1)}a_{ij_3}^{(3)}\right)^2+\sum_{j_2,j_3=1}^{3}\left(a_{ij_2}^{(2)}a_{ij_3}^{(3)}\right)^2 \\ & +\sum_{j_1,j_2,j_3=1}^{3}\left(a_{ij_1}^{(1)}a_{ij_2}^{(2)}a_{ij_3}^{(3)}\right)^2\Bigg]|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}| \\ \le & \frac{1}{8}\sum_i p_i\frac{1}{2}(1+C_3^1\times 1+C_3^2\times 1 \\ & +C_3^3\times 1)|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}| \\ = & 4\times\frac{1}{8}\sum_i p_i|\omega_i^{A_4}\rangle\langle\omega_i^{A_4}|. \end{aligned} \tag{9}$$

This means that $4L_0-\sum_{j=1}^{63} l_jL_j$ is positive. QED.

Theorem 2 If $\rho_{A_1A_2A_3A_4}$ is separable, then $\mathrm{tr}\left(7L_0^2-\sum_{j=1}^{63}L_j^2\right)\ge 0$.

Proof From Eqs. (6), the inequality

$$\sum_{j_r=1}^{3} a_{ij_r}^{(r)}a_{kj_r}^{(r)} \le \frac{1}{2}(1+1)=1 \tag{10}$$

can be obtained. And from Eqs.(7) we can obtain

$$\begin{aligned} & \mathrm{tr}(7L_0^2-\sum_{j=1}^{63}M_j^2) \\ = & \frac{1}{64}\sum_{ik}p_ip_k\Bigg[7-\Bigg(\sum_{r=1}^{3}\sum_{j_r=1}^{3}a_{ij_r}^{(r)}a_{kj_r}^{(r)} \\ & +\sum_{j_1,j_2=1}^{3}a_{ij_1}^{(1)}a_{kj_1}^{(1)}a_{ij_2}^{(2)}a_{kj_2}^{(2)} \\ & +\sum_{j_1,j_3=1}^{3}a_{ij_1}^{(1)}a_{kj_1}^{(1)}a_{ij_3}^{(3)}a_{kj_3}^{(3)} \\ & +\sum_{j_2,j_3=1}^{3}a_{ij_2}^{(2)}a_{kj_2}^{(2)}a_{ij_3}^{(3)}a_{kj_3}^{(3)} \\ & +\sum_{j_1,j_2,j_3=1}^{3}a_{ij_1}^{(1)}a_{kj_1}^{(1)}a_{ij_2}^{(2)}a_{kj_2}^{(2)}a_{ij_3}^{(3)}a_{kj_3}^{(3)}\Bigg)\Bigg]|\langle\omega_i^{A_4}|\omega_k^{A_4}\rangle|^2 \\ \ge & \frac{1}{64}\sum_{ik}p_ip_k[7-(C_3^1\times 1+C_3^2\times 1 \\ & +C_3^3\times 1)]|\langle\omega_i^{A_4}|\omega_k^{A_4}\rangle|^2 \\ \ge & 0. \quad \text{QED.} \end{aligned} \tag{11}$$

Now we generalize the results to $2\times 2\times\cdots\times 2\times N$ quantum systems.

Let the dimensions of systems $A_1, A_2, \ldots, A_n$ and A_{n+1} be 2, 2,..., 2 and N, respectively. A given state $\rho_{A_1A_2\ldots A_{n+1}}$ can be written as

$$\begin{aligned} & \rho_{A_1A_2\ldots A_{n+1}} \\ = & I_2\otimes\cdots\otimes I_2\otimes L_0+\sum_{r=1}^{n}\sum_{j_r=1}^{3} I_2\otimes\cdots \\ & \otimes\lambda_{j_r}^{(r)}\otimes\cdots\otimes I_2\otimes L_{j_r}^{(r)}+\cdots \\ & +\sum_{j_1,\ldots,j_n=1}^{3}\lambda_{j_1}^{(1)}\otimes\cdots\otimes\lambda_{j_n}^{(n)}\otimes L_{j_1\ldots j_n}^{(1\cdots n)}, \end{aligned} \tag{12}$$

where $\lambda_{j_r}^{(r)}$ $(j_r=1,2,3; r=1,\ldots,n)$ are Pauli matrices, $L_0,\ldots,L_{j_r}^{(r)},\ldots,L_{j_1\ldots j_n}^{(1\cdots n)}$, $(j_1,\ldots,j_n=1,2,3)$ are N-dimensional Hermitian matrices. A separable state

$$\rho_{A_1A_2\ldots A_{n+1}}=\sum_i p_i|\psi_i^{A_1}\rangle\langle\psi_i^{A_1}|\otimes\cdots\otimes|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \tag{13}$$

can be also written as

$$\begin{aligned} & \rho_{A_1A_2\ldots A_{n+1}} \\ = & \sum_i p_i\frac{1}{2}\left(I_2+\sum_{j_1=1}^{3}a_{ij_1}^{(1)}\lambda_{j_1}^{(1)}\right)\otimes\cdots \\ & \otimes\frac{1}{2}\left(I_2+\sum_{j_n=1}^{3}a_{ij_n}^{(n)}\lambda_{j_n}^{(n)}\right)\otimes|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|, \end{aligned} \tag{14}$$

where $(a_{i1}^{(1)}, a_{i2}^{(1)}, a_{i3}^{(1)})$, $\cdots$, $(a_{i1}^{(n)}, a_{i2}^{(n)}, a_{i3}^{(n)})$ are real vectors on the Bloch sphere and satisfy

$$\sum_{j_1=1}^{3}\left(a_{ij_1}^{(1)}\right)^2=1,\ \cdots,\ \sum_{j_n=1}^{3}\left(a_{ij_n}^{(n)}\right)^2=1. \tag{15}$$

Comparing Eqs.(12) and (14), we have

$$\begin{aligned} & L_0=\frac{1}{2^n}\sum_i p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|, \\ & L_{j_1}^{(1)}=\frac{1}{2^n}\sum_i a_{ij_1}^{(1)}p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|, \\ & \cdots\cdots, \\ & L_{j_1\ldots j_n}^{(1\cdots n)}=\frac{1}{2^n}\sum_i a_{ij_1}^{(1)}\cdots a_{ij_n}^{(n)}p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|. \end{aligned} \tag{16}$$

Now we deal with the separability in a way similar to that for Theorems 1 and 2. Let $\sum_{j=1}^{4^n-1} l_jL_j$ and vector $(l_1, l_2\ldots, l_{4^n-1})$ denote $\sum_{r=1}^{n}\sum_{j_r=1}^{3} l_{j_r}^{(r)}L_{j_r}^{(r)}+$

$\cdots+\sum_{j_1,\ldots,j_n=1}^{3} l_{j_1\cdots j_n}^{(1\cdots n)} L_{j_1\ldots j_n}^{(1\cdots n)}$ and $(l_1^{(1)}, l_2^{(1)}, \ldots, l_{3\ldots3}^{(1\cdots n)})$ respectively, then we have the following theorem.

Theorem 3 If $\rho_{A_1A_2\ldots A_{n+1}}$ is separable, then

(1) $2^{n-1}L_0 - \sum_{j=1}^{4^n-1} l_j L_j$ is positive for any vector $(l_1, l_2 \ldots, l_{4^n-1})$ of unit length or less;

(2) $\mathrm{tr}((2^n-1)L_0^2 - \sum_{j=1}^{4^n-1} L_j^2) \geq 0$.

Proof Applying Eqs.(16) and the assumption of the theorem, we can obtain

$$\begin{aligned}\sum_{j=1}^{4^n-1} l_j L_j \leq & \frac{1}{2^n}\sum_i p_i \frac{1}{2}\Bigg[\sum_{j=1}^{4^n-1}(l_j)^2 \\ & + \sum_{r=1}^{n}\sum_{j_r=1}^{3}\left(a_{ij_r}^{(r)}\right)^2 + \cdots \\ & + \sum_{j_1,\ldots,j_n=1}^{3}\left(a_{ij_1}^{(1)}\cdots a_{ij_n}^{(n)}\right)^2\Bigg]|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \\ \leq & \frac{1}{2^n}\sum_i p_i \frac{1}{2}(1 + C_n^1\times 1 + C_n^2\times 1 + \cdots \\ & + C_n^n \times 1)|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \\ = & 2^{n-1}\times\frac{1}{2^n}\sum_i p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|. \end{aligned} \tag{17}$$

In the last step the equality

$$\begin{aligned} & C_n^1 + C_n^2 + \cdots + C_n^n \\ = & (C_n^0 + C_n^1 + C_n^2 + \cdots + C_n^n) - C_n^0 = 2^n - 1 \end{aligned} \tag{18}$$

has been used. We can also have

$$\begin{aligned} & \mathrm{tr}((2^n-1)L_0^2 - \sum_{j=1}^{4^n-1} L_j^2) \\ = & \frac{1}{4^n}\sum_{ik} p_i p_k\Bigg[2^n - 1 - \Bigg(\sum_{r=1}^{n}\sum_{j_r=1}^{3} a_{ij_r}^{(r)} a_{kj_r}^{(r)} \\ & + \sum_{j_1,j_2=1}^{3} a_{ij_1}^{(1)} a_{kj_1}^{(1)} a_{ij_2}^{(2)} a_{kj_2}^{(2)} + \cdots \\ & + \sum_{j_1,\ldots,j_n=1}^{3} a_{ij_1}^{(1)} a_{kj_1}^{(1)}\cdots a_{ij_n}^{(n)} a_{kj_n}^{(n)}\Bigg)\Bigg]|\langle\mu_i^{A_{n+1}}|\mu_k^{A_{n+1}}\rangle|^2 \\ \geq & \frac{1}{4^n}\sum_{ik} p_i p_k[2^n - 1 - (C_n^1\times 1 + \cdots \\ & + C_n^n\times 1)]|\langle\mu_i^{A_{n+1}}|\mu_k^{A_{n+1}}\rangle|^2 \\ \geq & 0. \end{aligned} \tag{19}$$

The theorem follows from Eqs.(17) and (19). QED.

3. Separability in $R\times R\times\cdots\times R\times N$ quantum systems

We begin with the following lemma.

Lemma[19] The density matrix of a pure state is given by

$$\rho = \frac{1}{2}\left(\frac{2}{R}I_R + \sum_{j_r=1}^{R^2-1} a_{j_r}^{(r)}\lambda_{j_r}^{(r)}\right), \tag{20}$$

where I_R is the R-dimensional identity matrix, $a_{j_r}^{(r)}$ is the expectation value of $\lambda_{j_r}^{(r)}$ and $\lambda_{j_r}^{(r)}$ $(j_r = 1, 2, \ldots, R^2-1)$ are generators of $SU(R)$, then

$$\sum_{j_r=1}^{R^2-1}\left(a_{j_r}^{(r)}\right)^2 = 2(1-\frac{1}{R}). \quad \text{QED.} \tag{21}$$

Applying the above lemma, the inequality

$$\begin{aligned} & \sum_{j_r=1}^{R} a_{ij_r}^{(r)} a_{kj_r}^{(r)} \leq \frac{1}{2}\left(2\left(1-\frac{1}{R}\right) + 2\left(1-\frac{1}{R}\right)\right) \\ = & 2\left(1-\frac{1}{R}\right) \end{aligned} \tag{22}$$

can be obtained.

Let the dimensions of systems A_1, A_2, ..., A_n and A_{n+1} be R, R,..., R and N, respectively. A given state $\rho_{A_1A_2\ldots A_{n+1}}$ can be written as

$$\begin{aligned} & \rho_{A_1A_2\ldots A_{n+1}} \\ = & I_R\otimes\cdots\otimes I_R\otimes L_0 + \sum_{r=1}^{n}\sum_{j_r=1}^{R^2-1} I_R\otimes\cdots \\ & \otimes\lambda_{j_r}^{(r)}\otimes\cdots\otimes I_R\otimes L_{j_r}^{(r)} + \cdots \\ & + \sum_{j_1,\ldots,j_n=1}^{R^2-1}\lambda_{j_1}^{(1)}\otimes\cdots\otimes\lambda_{j_n}^{(n)}\otimes L_{j_1\ldots j_n}^{(1\cdots n)}, \end{aligned} \tag{23}$$

where $\lambda_{j_r}^{(r)}$ $(j_r = 1, 2, \cdots, R^2-1; r = 1, \ldots, n)$ are generators of $SU(R)$, and $L_0, \ldots, L_{j_r}^{(r)}, \ldots, L_{j_1\ldots j_n}^{(1\cdots n)}$ $(j_1, \ldots, j_n = 1, 2, \cdots, R^2-1)$ are N-dimensional Hermitian matrices. A separable state

$$\rho_{A_1A_2\ldots A_{n+1}} = \sum_i p_i|\psi_i^{A_1}\rangle\langle\psi_i^{A_1}|\otimes\cdots\otimes|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \tag{24}$$

can be also written as

$$\begin{aligned} & \rho_{A_1A_2\ldots A_{n+1}} \\ = & \sum_i p_i\frac{1}{2}\left(\frac{2}{R}I_R + \sum_{j_1=1}^{R^2-1} a_{ij_1}^{(1)}\lambda_{j_1}^{(1)}\right)\otimes\cdots \\ & \otimes\frac{1}{2}\left(\frac{2}{R}I_2 + \sum_{j_n=1}^{R^2-1} a_{ij_n}^{(n)}\lambda_{j_n}^{(n)}\right)\otimes|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|. \end{aligned} \tag{25}$$

Comparing Eqs.(23) and (25), we have

$$L_0 = \frac{1}{R^n}\sum_i p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|,$$

$$L_{j_1}^{(1)} = \frac{1}{2R^{n-1}}\sum_i a_{ij_1}^{(1)} p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|,$$

$$\cdots\cdots,$$

$$L_{j_1\ldots j_n}^{(1\cdots n)} = \frac{1}{2^n}\sum_i a_{ij_1}^{(1)}\cdots a_{ij_n}^{(n)} p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|. \quad (26)$$

Let $\sum_{j=1}^{R^{2n}-1} l_j L_j$ and vector $(l_1, l_2, \ldots, l_{(R^{2n}-1)})$ denote $\sum_{r=1}^{n}\sum_{j_r=1}^{R^2-1} l_{j_r}^{(r)} L_{j_r}^{(r)} + \cdots + \sum_{j_1,\ldots,j_n=1}^{R^2-1} l_{j_1\cdots j_n}^{(1\cdots n)} L_{j_1\ldots j_n}^{(1\cdots n)}$ and $(l_1^{(1)}, l_2^{(1)}, \ldots, l_{(R^2-1)\ldots(R^2-1)}^{(1\cdots n)})$ respectively, we will have the following theorem.

Theorem 4 If $\rho_{A_1A_2\ldots A_{n+1}}$ is separable, then

(1) $\frac{1}{2}\left(1+\frac{R(R-1)}{2}\right)^n L_0 - \sum_{j=1}^{R^{2n}-1} l_j L_j$ is positive for any vector $(l_1, l_2, \ldots, l_{(R^{2n}-1)})$ of unit length or less;

(2) $\mathrm{tr}\left(\left[\left(1+\frac{R(R-1)}{2}\right)^n - 1\right]L_0^2 - \sum_{j=1}^{R^{2n}-1} L_j^2\right) \geq 0$.

Proof Applying (22) and the assumption of the theorem, we can obtain

$$\begin{aligned}
&\sum_{j=1}^{R^{2n}-1} d_j L_j \\
\leq &\sum_i p_i \frac{1}{R^n}\Bigg[\sum_{j=1}^{R^{2n}-1}(d_j)^2 + \sum_{r=1}^{n}\sum_{j_r=1}^{R^2-1}\frac{R^2}{2^2}(a_{ij_r}^{(r)})^2 + \cdots \\
&+ \sum_{j_1,\ldots,j_n=1}^{R^2-1}\frac{R^{2n}}{2^{2n}}(a_{ij_1}^{(1)}\cdots a_{ij_n}^{(n)})^2\Bigg]|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \\
\leq &\frac{1}{R^n}\sum_i p_i \frac{1}{2}(1 + \frac{R(R-1)}{2}C_n^1 + \cdots \\
&+ \left(\frac{R(R-1)}{2}\right)^n C_n^n)|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}| \\
= &\frac{1}{2}\left(1+\frac{R(R-1)}{2}\right)^n \frac{1}{R^n}\sum_i p_i|\mu_i^{A_{n+1}}\rangle\langle\mu_i^{A_{n+1}}|, \quad (27)
\end{aligned}$$

and $\frac{1}{2}\left(1+\frac{R(R-1)}{2}\right)^n L_0 - \sum_{j=1}^{R^{2n}-1} l_j L_j$ is positive. We can also obtain

$$\begin{aligned}
&\mathrm{tr}\left(\left[\left(1+\frac{R(R-1)}{2}\right)^n - 1\right]L_0^2 - \sum_{j=1}^{R^{2n}-1} L_j^2\right) \\
= &\frac{1}{R^{2n}}\sum_{ik} p_i p_k \Bigg[\left(1+\frac{R(R-1)}{2}\right)^n - 1 \\
&- \Bigg(\sum_{r=1}^{n}\sum_{j_r=1}^{R^2-1}\frac{R^2}{2^2}a_{ij_r}^{(r)}a_{kj_r}^{(r)} + \cdots \\
&+ \sum_{j_1,\ldots,j_n=1}^{R^2-1}\frac{R^{2n}}{2^{2n}}a_{ij_1}^{(1)}a_{kj_1}^{(1)}\cdots a_{ij_n}^{(n)}a_{kj_n}^{(n)}\Bigg)\Bigg] \\
&\times |\langle\mu_i^{A_{n+1}}|\mu_k^{A_{n+1}}\rangle|^2 \\
\geq &\frac{1}{R^{2n}}\sum_{ik} p_i p_k \Bigg[\left(1+\frac{R(R-1)}{2}\right)^n - 1 \\
&- \Bigg(\frac{R(R-1)}{2}C_n^1 + \frac{R^2(R-1)^2}{2^2}C_n^2 + \cdots \\
&+ \left(\frac{R(R-1)}{2}\right)^n C_n^n\Bigg)\Bigg]|\langle\mu_i^{A_{n+1}}|\mu_k^{A_{n+1}}\rangle|^2 \\
\geq &0. \quad \text{QED.} \quad (28)
\end{aligned}$$

4. Conclusion

We have investigated the separability of mixed states in multipartite quantum systems. Some quantity in relation to the Hermitian matrix is positive if a quantum state in $2\times 2\times\cdots\times 2\times N$ (or $R\times R\times\cdots\times R\times N$) quantum systems is separable.

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