Vol 14 No 2, February 2005
1009-1963/2005/14(02)/0247-04
Chinese Physics
© 2005 Chin. Phys. Soc.
and IOP Publishing Ltd

The multi-component Tu hierarchy of soliton equations and its multi-component integrable couplings system*

Xia Tie-Cheng(夏铁成)[a)b)†], Wang Hong(汪 宏)[c)], and Zhang Yu-Feng(张玉峰)[d)]

[a)] *Department of Mathematics, Tianjin University, Tianjin* 300072, *China*
[b)] *Department of Mathematics, Bohai University, Jinzhou* 121000, *China*
[c)] *Department of Physics, Tianjin University, Tianjin* 300072, *China*
[d)] *Department of Mathematics, Liaoning Normal University, Dalian* 116022, *China*

(Received 8 October 2003; revised manuscript received 27 September 2004)

A new simple loop algebra $\widetilde{G}_M$ is constructed, which is devoted to the establishing of an isospectral problem. By making use of the Tu scheme, the multi-component Tu hierarchy is obtained. Furthermore, an expanding loop algebra $\widetilde{F}_M$ of the loop algebra $\widetilde{G}_M$ is presented. Based on the $\widetilde{F}_M$, the multi-component integrable coupling system of the multi-component Tu hierarchy has been worked out. The method can be applied to other nonlinear evolution equation hierarchies.

Keywords: loop algebra, Tu hierarchy, integrable couplings system
PACC: 0340K, 0220, 0365G

1. Introduction

Searching for new integrable hierarchies of soliton equations is an important and interesting topic in soliton theory. Various efficient approaches have been taken to obtaining many integrable systems such as the AKNS hierarchy, KN hierarchy, Schrödinger system, and so on.[1−38] Multi-component integrable hierarchies have been developed in Refs.[24,25]. In Refs.[26–28], a simple and efficient method for generating multi-component integrable hierarchies was proposed. Constructing a new simple loop algebra becomes a key step in this method. Although using the loop algebra in Refs.[26,27] can produce many multi-component integrable hierarchies, it is not suitable for seeking the multi-component Tu hierarchy.[29] In this paper, a new loop algebra $\widetilde{G}_M$ is first constructed, and then an isospectral problem is designed. By employing the Tu scheme, the multi-component Tu hierarchy is worked out. In addition, an expanding loop algebra $\widetilde{F}_M$ of the loop algebra $\widetilde{G}_M$ is presented, which is devoted to the deducing of the integrable coupling of the Tu hierarchy.

In 1989, Tu investigated the eigenvalue problem

$$\phi_x = U(u,\lambda)\phi,$$

$$U = \begin{pmatrix} -\lambda + \frac{1}{2}q & r \\ r & \lambda - \frac{1}{2}q \end{pmatrix},$$

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2, \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r, \end{pmatrix},$$

and obtained the so-called Tu hierarchy of evolution equations:[29]

$$u_{tn} = \begin{pmatrix} q \\ r \end{pmatrix}_{tn} = JL^{n-1}\begin{pmatrix} -\frac{1}{2}r^2 \\ -rq \end{pmatrix}$$

$$= JL^n\begin{pmatrix} 0 \\ -r \end{pmatrix} = J\frac{\delta H_{n+1}}{\delta u}, \quad n = 1,2,\cdots, \qquad (1)$$

where operator J and Hamiltonian H_n were given by

$$J = \begin{pmatrix} 0 & \partial r^{-1} \\ r^{-1}\partial & 0 \end{pmatrix}, \quad H_n = \frac{2}{n}a_{n+1},$$

*Projet supported by the National Natural Science Foundation of China (Grant No 10371070), the Special Funds for Major Specialities of Shanghai Educational Committee and Natural Science Foundation of Educational Committee of Liaoning Province. (Grant No 2004C057).
†E-mail: xiatc@yahoo.com.cn

$$\frac{\delta H_n}{\delta u} = \begin{pmatrix} a_n \\ b_n + c_n \end{pmatrix}.$$

In this paper we would like to extend Eq.(1) to a generalized Tu hierarchy: namely, the multi-component Tu hierarchy. Finally, the multi-component integrable couplings of the multi-component Tu hierarchy are obtained.

2. A new loop algebra

If $G_M = \{a = (a_{ij})_{M\times 3} = (a_1, a_2, a_3)\}$ denotes a set of matrices, where M is a positive integer, $a_i (i = 1, 2, 3)$ is the ith column of the matrix a. Then $\widetilde{G}_M$ is a linear space.

Let

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}}, \quad \beta = (\beta_1, \beta_2, \ldots, \beta_M)^{\mathrm{T}},$$

and define their product as $\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, \cdots, \alpha_M\beta_M)^{\mathrm{T}}$. If $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in G_M$, a commutation operation for G_M is defined as

$$\begin{aligned}[a, b] =& (a_3 * b_2 - a_2 * b_3, a_1 * b_3 \\ & - a_3 * b_1, a_1 * b_2 - a_2 * b_1),\end{aligned} \tag{2}$$

for $\forall\ a, b, c \in \tilde{G}_M$, we can verify that

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \tag{3}$$

i.e. Jacobian identity holds. Therefore, G_M with the operation (1) becomes the Lie algebra. A corresponding loop algebra $\widetilde{G}_M$ is defined as

$$\widetilde{G}_M = \{a\lambda^n, a \in G_M, n = 0, \pm 1, \pm 2, \ldots\}, \tag{4}$$

with a commutation operation defined as

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \forall a, b \in G_M. \tag{5}$$

Since $\widetilde{G}_1$ is equal to $\widetilde{A}_1$ in Refs.[30–32], we conclude that $\widetilde{G}_M$ is an extension of the loop algebra $\widetilde{A}_1$. We also find $\widetilde{G}_M$ has two features: (i) the commutation operation is simple and straightforward, as that in the loop algebra $\widetilde{A}_1$ and (ii) by means of $\widetilde{G}_M$, we can proceed to do simple calculations to be able to obtain various multi-component integrable systems.

Consider a linear isospectral problems as follows:

$$\begin{cases} \phi_x = [U, \phi], \lambda_t = 0, \phi, U, V \in \widetilde{G}_M \\ \phi_t = [V, \phi], \end{cases} \tag{6}$$

whose compatibility gives rise to

$$\phi_{xt} = [U_t, \phi] + [U, [V, \phi]] = \phi_{tx} = [V_x, \phi] + [V, [U, \phi]],$$

that is,

$$[U_t, \phi] - [V_x, \phi] + [U, [V, \phi]] + [V, [U, \phi]] = 0. \tag{7}$$

By employing the definition (3), the formula (7) can be written as

$$[U_t, \phi] - [V_x, \phi] + [[U, V], \phi] = 0. \tag{8}$$

Since ϕ is arbitrary, a condition of Eq.(8) holds if and only if the following equation exists:

$$U_t - V_x + [U, V] = 0. \tag{9}$$

Hence, the compatibility of the formula (6) leads to the zero-curvature equation (9). This indicates that the soliton equations derived from the Lax pair (6) are Lax integrable.

3. The multi-component Tu hierarchy

We consider an isospectral problem

$$\phi_x = [U, \phi], \lambda_t = 0, U = (-2\lambda I_M + q * I_M, 2r * I_M, 0), \tag{10}$$

where $I_M = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{M\times 1}$, $q = (q_1, q_2, \ldots, q_M)^{\mathrm{T}}$, and $r = (r_1, r_2, \ldots, r_M)^{\mathrm{T}}$.

Let

$$V = \sum_{m=0}^{\infty} (a_m, b_m, c_m)\lambda^{-2m},$$

where

$$a_m = (a_{m1}, a_{m2}, \ldots, a_{mM})^{\mathrm{T}},$$

$$b_m = (b_{m1}, b_{m2}, \ldots, b_{mM})^{\mathrm{T}},$$

and

$$c_m = (c_{m1}, c_{m2}, \ldots, c_{mM})^{\mathrm{T}}.$$

Solving the stationary zero curvature equation

$$V_x = [U, V] \tag{11}$$

gives

$$a_{mx} = -2r * c_m, \tag{12a}$$

$$b_{mx} = -2c_{m+1} + q * c_m, \tag{12b}$$

$$c_{mx} = -2b_{m+1} + q * b_m - 2r * a_m, m \geq 0, \tag{12c}$$

$$b_0 = c_0 = 0, a_0 = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}} = \text{const.}, \tag{12d}$$

$$a_1 = c_1 = 0, b_1 = -r. \tag{12e}$$

Denoting

$$V_+^{(n)} = \sum_{m=0}^{n} (a_m, b_m, c_m)\lambda^{n-m},$$

$$V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

Eq.(11) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{13}$$

Again taking $\Delta_n = (-r^{-1} * b_{n+1}, 0, 0)$, and $V^{(n)} = V_+^{(n)} + \Delta_n$, here without loss of generality, and supposing that vector r^{-1} is defined as $r * r^{-1} = r^{-1} * r = I_M$, the zero curvature equation

$$U_t - V^{(n)} + [U, V^{(n)}] = 0, \tag{14}$$

gives rises to the Lax integrable system

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = JL^n \begin{pmatrix} 0 \\ -r \end{pmatrix}, \tag{15}$$

with

$$J = \begin{pmatrix} 0 & \partial r^{-1} * \\ r^{-1} * \partial & 0 \end{pmatrix},$$

and

$$L = \begin{pmatrix} \frac{1}{2} q * \partial & \frac{1}{2} r * \partial \\ \frac{1}{2} \partial r^{-1} * \partial & \frac{1}{2} q * \end{pmatrix}.$$

When $M = 1$, the system (15) is just the Tu hierarchy.[29] When $M > 1$, the system (15) is the multi-component Tu hierarchy.

4. The multi-component integrable coupling system

Set

$$F_M = \{a = (a_{ij})_{M\times 5} = (a_1, a_2, a_3, a_4, a_5)\}, \tag{16}$$

with a commutation operation defined as

$$\begin{aligned}[a, b] =& (a_3 * b_2 - a_2 * b_3, a_1 * b_3 \\ & - a_3 * b_1, a_1 * b_2 - a_2 * b_1, \\ & \frac{1}{2}(a_1 * b_5 - a_5 * b_1 + a_2 * b_4 \\ & - a_4 * b_2 + a_5 * b_3 - a_3 * b_5), \\ & \frac{1}{2}(a_1 * b_4 - a_4 * b_1 + a_3 * b_4 \\ & - a_4 * b_3 + a_5 * b_2 - a_2 * b_5)). \end{aligned} \tag{17}$$

Then F_M is the Lie algebra. A corresponding loop algebra $\widetilde{F}_M$ is defined as

$$\widetilde{F}_M = \{a\lambda^n, a \in F_M, n = 0, \pm 1, \pm 2, \dots\} \tag{18}$$

with a commutation operation defined as

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \qquad \forall a, b \in G_M. \tag{19}$$

Denoting

$$\tilde{F}_M(1) = \{(a_1, a_2, a_3, 0, 0)\lambda^n\},$$

$$\tilde{F}_M(2) = \{(0, 0, 0, a_4, a_5)\lambda^n\},$$

yields (i) $\widetilde{F}_M = \tilde{F}_M(1) \oplus \tilde{F}_M(2)$, $\widetilde{F}_M(1) \cong \tilde{G}_M$; (ii) $[\widetilde{F}_M(1), \widetilde{F}_M(2)] \subset \widetilde{F}_M(2)$. Based on the loop algebra $\widetilde{F}_M$, we consider an isopectral problem

$$\begin{aligned} \phi_x =& [U, \phi], \\ U =& (-2\lambda I_M + u_1 * I_M, 2u_2 * I_M, 0, \\ & u_3 * I_M, u_4 * I_M), \end{aligned} \tag{20}$$

where $u_i = (u_{i1}, u_{i2}, \dots, u_{iM})^{\mathrm{T}}, i = 1, 2, 3, 4$. Let $V = \sum_{i=0}^{n} (a_m, b_m, c_m, d_m, f_m)\lambda^{-2m}$. Solving the equation similar to Eq.(11) gives

$$a_{mx} = -2u_2 * c_m, \tag{21a}$$

$$b_{mx} = -2c_{m+1} + u_1 * c_m, \tag{21b}$$

$$c_{mx} = -2b_{m+1} + u_1 * b_m - 2u_2 * a_m, \tag{21c}$$

$$\begin{aligned} f_{m+1} =& -d_{mx} + \frac{1}{2}u_1 * f_m + u_2 * d_m - \frac{1}{2}u_3 * b_m \\ & - \frac{1}{2}u_4 * a_m + \frac{1}{2}u_4 * c_m, \end{aligned} \tag{21d}$$

$$\begin{aligned} d_{m+1} =& -f_{mx} + \frac{1}{2}u_1 * d_m - u_2 * f_m - \frac{1}{2}u_3 * a_m \\ & - \frac{1}{2}u_3 * c_m + \frac{1}{2}u_4 * b_m, \end{aligned} \tag{21e}$$

$$\begin{aligned} & b_0 = c_0 = d_0 = f_0 = 0, \\ & a_0 = \alpha = (\alpha_1, \dots, \alpha_M) = \text{const.vector}, \end{aligned} \tag{21f}$$

$$\begin{aligned} & a_1 = c_1 = 0, b_1 = -u_2, \\ & d_1 = -\frac{\alpha}{2} * u_3, f_1 = -\frac{\alpha}{2} * u_4. \end{aligned} \tag{21g}$$

Taking $\Delta_n = (u_2^{-1} * b_{n+1}, 0, 0)$, $V^{(n)} = V_+^{(n)} + \Delta_n$, the zero curvature equation similar to Eq.(14) has the following form:

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4) \end{pmatrix}_t = \begin{pmatrix} 0 & \partial u_2^{-1}* & 0 & 0 \\ u_2^{-1}*\partial & 0 & 0 & 0 \\ 0 & \frac{1}{2}u_2^{-1}*u_4 & 0 & -I_M \\ 0 & \frac{1}{2}u_2^{-1}*u_3 & -I_M & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}a_{n+1} \\ b_{n+1} \\ d_{n+1} \\ f_{n+1} \end{pmatrix} = J \begin{pmatrix} \frac{1}{2}a_{n+1} \\ b_{n+1} \\ d_{n+1} \\ f_{n+1} \end{pmatrix}. \tag{22}$$

Here, without loss of generality, we also suppose that the vector u_2^{-1} satisfies $u_2 * u_2^{-1} = u_2^{-1} * u_2 = I_M$. From the definition (21), a recurrence operator is presented in the form

$$L = \begin{pmatrix} \frac{1}{2}u_1*\partial & \frac{1}{2}u_2*\partial & 0 & 0 \\ \frac{1}{2}\partial u_2^{-1}*\partial & \frac{1}{2}u_1* & 0 & 0 \\ -u_3*+\frac{1}{2}u_2^{-1}*u_3*\partial & \frac{1}{2}u_4* & \frac{1}{2}u_1* & -\partial - u_2* \\ -u_4*-\frac{1}{2}u_2^{-1}*u_4*\partial & -\frac{1}{2}u_3* & u_2*-\partial & \frac{1}{2}u_1* \end{pmatrix}.$$

Thus, Eq.(22) can be written as

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = JL^n \begin{pmatrix} 0 \\ -u_2 \\ -\frac{\alpha}{2}*u_3 \\ -\frac{\alpha}{2}*u_4 \end{pmatrix}. \tag{23}$$

When $M = 1$, the system (23) is the integrable coupling of the system (15).[29] When $M > 1$ the system (23) is the multi-component integrable coupling of the system (15).

Remarks: The loop algebra presented here can be used for other known integrable hierarchies of soliton equations for generating the multi-component systems. But there exists an open problem. How do we improve the loop algebra $\widetilde{G}_M$ to work out the Hamiltonian structures, infinite conserved laws of the multi-component hierarchies? This is worth studying in the future.

References

[1] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
[2] Newell A C 1985 *Soliton in Mathematics and Physics* (Philadelphia, PA: SIAM)
[3] Wadati M 1972 *J. Phys. Soc. Jan.* **32** 1681
[4] Wadati M 1973 *J. Phys. Soc. Jan.* **34** 1289
[5] Wadati M Sanuki H and Konno K 1975 *Prog. Theor. Phys.* **53** 419
[6] Tu G Z 1989 *J. Math. Phys.* **30** 330
[7] Fan E G 2001 *J. Phys. A: Math. Gen.* **34** 513
[8] Xia T C, Chen X H and Chen D Y 2004 *Chaos, Solitons Fractals* **22** 939
[9] Xia T C, Chen X H, Chen D Y and Zhang Y F 2004 *Commun. Theor.Phys.* (Beijing China) **42** 180
[10] Fan E G 2001 *Physica* A **301** 105
[11] Fan E G 2000 *J. Math. Phys.* **41** 7769
[12] Tsuchida T and Wadati M 1998 *Chaos Solitons Fractals* **9** 869
[13] Ma W X 1996 *Chaos Solitons Fractals* **7** 1227
[14] Ma W X 1992 *J. Math. Ann.* **13** 115 (in Chinese)
[15] Ma W X and Zhou R G 1999 *J. Math. Phys.* **40** 4419
[16] Zhang Y F 2003 *Chaos Solitons Fractals* **18** 855
[17] Zhang Y F and Yan Q Y 2004 *Chaos Solitons Fractals* **21** 413
[18] Xia T C, Yu F J and Chen D Y 2004 *Physica* A **343** 238
[19] Cao C W 1990 *Sci. China. Ser.* A **33** 528
[20] Geng X G 1993 *J. Math. Phys.* **34** 805
[21] Zeng Y B 1991 *Phys. Lett.* A **160** 541
[22] Zeng Y B and Hietarinta J 1996 *J. Phys. A Math. Gen.* **29** 5241
[23] Xia T C Chen X H and Chen D Y 2005 *Chaos Solitons Fractals* **23** 1033
[24] Tsuchida T and Wadati M 1999 *J. Phys. Lett.* A **257** 53
[25] Tsuchida T and Wadati M 1999 *J. Phys. Soc. Jan.* **69** 2241
[26] Guo F K and Zhang Y F 2003 *J. Math. Phys.* **44** 5793
[27] Zhang Y F 2004 *Chaos, Solitons Fractals* **21** 305
[28] Xia T C, Yu F J and Chen D Y 2004 *Commun. Theor. Phys.* (Beijing, China) **42** 494
[29] Tu G Z and Meng D Z 1989 *Acta Math. Appl. Sin.* **5** 89
[30] Guo F K 2000 *Acta Math. Appl. Sin.* **23** 181
[31] Guo F K 2003 *J. Syst. Sci. Math. Sci.* **22** 36
[32] Guo F K 1999 *Acta Math. Phys. Sin.* **19** 507
[33] Zhang Y F Yan Q Y and Zhang H Q 2003 *Acta Phys. Sin.* **52** 5 (in Chinese)
[34] Guo F K *et al* 2002 *Acta Phys. Sin.* **51** 951 (in Chinese)
[35] Zhang Y F 2003 *Acta Phys. Sin.* **52** 2190 (in Chinese)
[36] Zhang Y F 2003 *Chin. Phys.* **12** 1194
[37] Zhang D J 2002 *J. Phys. A Math. Gen.* **35** 7225
[38] Xia T C Yu F J and Chen D Y 2005 *Chaos, Solitons Fractals* **23** 1163