Vol 14 No 2, February 2005
1009-1963/2005/14(02)/0244-03
Chinese Physics
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The forms of three-order Lagrangian equation in relative motion

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(Received 4 August 2004; revised manuscript received 4 October 2004)

In this paper, the general expressions of three-order Lagrangian equations in a motional coordinate system are obtained. In coordinate systems with some specific forms of motion, the expressions corresponding to these equations are also presented.

Keywords: three order Lagrangian equations, relative motion, time rate of change of force
PACC: 0320

1. Introduction

In mechanical research, the motion problem of a mechanical system in different coordinate systems is a basic problem. Usually, in mechanical study, the motion equations describing the motion laws of a mechanical system in a stationary or a uniform motion coordinate system are needed. But for many practical problems, the process of finding these motion equations has to be finished in a non-inertial coordinate system in which the acceleration is not equal to zero. Especially, for some given basic motion equations, it is very important that how we can obtain their basic forms in a non-inertial coordinate system. In recent years, many theoretical researchers have been interested in studying some basic problems in analytical mechanics. Much work[1−25] has been done. In these researches, the basic theoretical problems of high velocity motion in different coordinate systems are emphasized. In 1985, Carmeli[1−4] developed the theory of the rotating relativistic mechanics. In 1996 Luo[5] established the theory of the rotating relativistic analytical mechanics, and deeply studied the relativistic analytical mechanics and rotating relativistic Birkhoff system[13−22] in many respects. In 2003, Jia[24] presented the theory of analytical static mechanics in the relativistic rotating system.

In this paper, based on the three-order Lagrangian equations given in Refs.[25] and [26], their forms in a non-inertial coordinate system under the non-relativistic condition are discussed. And the corresponding equations are also obtained.

2. The relative form of three-order Lagrangian equations

The three-order Lagrangian equations[25,26] are given as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_s}\right)-\frac{1}{2}\frac{\partial S}{\partial \dot{q}_s}=Q_s,\quad (s=1,2,\cdots,n),\tag{1}$$

$$S=\frac{1}{2}\sum_{i=1}^{N}m_i\ddot{\boldsymbol{R}}_i\cdot\ddot{\boldsymbol{R}}_i,\quad Q_s=\sum_{i=1}^{N}\dot{\boldsymbol{F}}_i\cdot\frac{\partial \boldsymbol{R}_i}{\partial q_s},$$

where S is the acceleration energy.

Let us consider a particle system consisting of N particles, a stationary coordinate system S $(O$–$XYZ)$ and a motional coordinate system S' $(o$–$xyz)$. The velocity of S' system relative to S system is $\boldsymbol{V}(t)$. The angular velocity of S' system is $\boldsymbol{\omega}(t)$. $\boldsymbol{r}_0(t)$ is the position vector of the coordinate origin o in the S' system relative to the coordinate origin O in the S system. In the particle system, the mass of the ith particle is m_i, the position vectors of the ith particle in the S system and the S' system are $\boldsymbol{R}_i$ and $\boldsymbol{r}_i$ respectively, and the $\boldsymbol{r}_i$ is not an explicit function of time t, i.e.

$$\boldsymbol{r}_i=\boldsymbol{r}_i(q_s),\quad (i=1,2,\cdots,N;s=1,2,\cdots,n).\tag{2}$$

From the relations of classical mechanics,[25] we

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have

$$\boldsymbol{R}_i = \boldsymbol{r}_0 + \boldsymbol{r}_i, \tag{3}$$

$$\boldsymbol{V}_i = \dot{\boldsymbol{R}}_i = \dot{\boldsymbol{r}}_0 + \dot{\boldsymbol{r}}_i^* + \boldsymbol{\omega} \times \boldsymbol{r}_i = \dot{\boldsymbol{r}}_0 + \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s} \dot{q}_s + \boldsymbol{\omega} \times \boldsymbol{r}_i, \tag{4}$$

$$\boldsymbol{a}_i = \ddot{\boldsymbol{R}}_i = \boldsymbol{a}_0 + \ddot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*, \tag{5}$$

where $\dot{\boldsymbol{r}}_i^*$ is the relative derivative of the position vector $\boldsymbol{r}_i$. This means that the differentiation calculation is performed just with respect to the coordinates of the vector $\boldsymbol{r}_i$, rather than the unit vector. Substituting $\ddot{\boldsymbol{R}}_i$ into the expression of acceleration energy S in Eq.(1) yields

$$\begin{aligned} S =& \frac{1}{2} \sum_{i=1}^{N} m_i [\boldsymbol{a}_0 + \ddot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i \\ &+ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*]^2 \\ =& S_1 + S_2 + S_3, \end{aligned} \tag{6}$$

$$S_1 = \frac{1}{2} \sum_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i^* \cdot \ddot{\boldsymbol{r}}_i^*,$$

$$\begin{aligned} S_2 =& \frac{1}{2} \sum_{i=1}^{N} m_i \{\boldsymbol{a}_0^2 + (\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i)^2 + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)]^2 \\ &+ 2\ddot{\boldsymbol{r}}_0 \cdot (\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i) \\ &+ 2\ddot{\boldsymbol{r}}_0 \cdot [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)] \\ &+ 2(\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i) \cdot [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)]\}, \end{aligned} \tag{8}$$

$$\begin{aligned} S_3 =& \frac{1}{2} \sum_{i=1}^{N} m_i \{4(\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*)^2 + 2\ddot{\boldsymbol{r}}_0 \cdot \ddot{\boldsymbol{r}}_i^* \\ &+ 4\ddot{\boldsymbol{r}}_0 \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*) \\ &+ 2\ddot{\boldsymbol{r}}_i^* \cdot (\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i) + 2\ddot{\boldsymbol{r}}_i^* \cdot [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)] \\ &+ 4\ddot{\boldsymbol{r}}_i^* \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*) \\ &+ 2(\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*) \\ &+ 2[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)](\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*)\}. \end{aligned} \tag{9}$$

Let operator E_s be

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \ddot{q}_s} - \frac{1}{2} \frac{\partial}{\partial \dot{q}_s} \tag{10}$$

and act on the acceleration energy S. Considering the form of Eq.(2), we obtain

$$E_s(S_1) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial S_1}{\partial \ddot{q}_s} - \frac{1}{2} \frac{\partial S_1}{\partial \dot{q}_s}, \tag{11}$$

$$E_s(S_2) = 0, \tag{12}$$

$$\begin{aligned} E_s(S_3) =& \frac{1}{2} \sum_{i=1}^{N} m_i \Bigg\{ [-4(\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*) - 2\ddot{\boldsymbol{r}}_0] \cdot \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \\ &- \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \cdot [2\ddot{\boldsymbol{r}}_i^* + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)] \\ &- \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \cdot (\dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i) \\ &+ \left(2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \ddot{q}_s} - \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \\ &\cdot [\ddot{\boldsymbol{r}}_0 + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*] \Bigg\}. \end{aligned} \tag{13}$$

Combining Eqs.(11) to (13) with Eq.(1), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial S_1}{\partial \ddot{q}_s} \right) - \frac{1}{2} \frac{\partial S_1}{\partial \dot{q}_s} \\ =& Q_s - E_s(S_3) \\ =& Q_s + M\ddot{\boldsymbol{r}}_0 \cdot \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_c^*}{\partial \dot{q}_s} \right) \\ &- \frac{1}{2} \sum_{i=1}^{N} m_i \Bigg\{ - \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \cdot [2\ddot{\boldsymbol{r}}_i^* + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)] \\ &- \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \cdot [4\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i] \\ &+ \left(2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \ddot{q}_s} - \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \\ &\cdot [\ddot{\boldsymbol{r}}_0 + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*] \Bigg\} \\ =& Q_s + M\ddot{\boldsymbol{r}}_0 \cdot \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_c^*}{\partial \dot{q}_s} \right) \\ &- \frac{1}{2} \sum_{i=1}^{N} m_i \Bigg\{ 2(\boldsymbol{\omega} \times \ddot{\boldsymbol{r}}_i^*) \cdot \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \\ &+ [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}_i)] \cdot \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \\ &- \left(\boldsymbol{\omega} \times \frac{\partial \dot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \cdot [4\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^* + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i] \\ &+ \left(2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \ddot{q}_s} - \frac{\partial \ddot{\boldsymbol{r}}_i^*}{\partial \dot{q}_s} \right) \\ &\cdot [\ddot{\boldsymbol{r}}_0 + \dot{\boldsymbol{\omega}}^* \times \boldsymbol{r}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) + 2\boldsymbol{\omega} \times \dot{\boldsymbol{r}}_i^*] \Bigg\}, \\ &(s = 1, 2, \cdots, n), \end{aligned} \tag{14}$$

where S_1 is the relative acceleration energy, M is the total particle mass of the particle system, $\dot{\boldsymbol{r}}_c$ is the velocity of the mass centre of the particle system, S_2 is the expression of acceleration energy in the motional coordinate system S'. The time rate of change of generalized force Q_s is given as follows:

$$Q_s = \sum_{i=1}^{N} \dot{\boldsymbol{F}}_i \cdot \frac{\partial (\boldsymbol{r}_0 + \boldsymbol{r}_i)}{\partial q_s} = \sum_{i=1}^{N} \dot{\boldsymbol{F}}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s}. \tag{15}$$

According to laws of vector calculation, we know that $\boldsymbol{\omega}\times\boldsymbol{r}_i$ is perpendicular to $\boldsymbol{r}_i$, and then Eq.(14) can be simplified as follows:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S_1}{\partial\ddot{q}_s}\right)-\frac{1}{2}\frac{\partial S_1}{\partial\dot{q}_s}\\ =&Q_s+M\ddot{\boldsymbol{r}}_0\cdot\left(\boldsymbol{\omega}\times\frac{\partial\dot{\boldsymbol{r}}_c^*}{\partial\dot{q}_s}\right)\\ &-\frac{1}{2}\sum_{i=1}^{N}m_i\left\{\left(2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\ddot{q}_s}-\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right)\right.\\ &\left.\cdot\left[\ddot{\boldsymbol{r}}_0+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_i)\right]\right\}\\ &+\frac{1}{2}\sum_{i=1}^{N}m_i\left\{\left(\boldsymbol{\omega}\times\frac{\partial\dot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right)\right.\\ &\left.\cdot\left[4\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_i^*+\dot{\boldsymbol{\omega}}^*\times\boldsymbol{r}_i\right]\right\},\quad(s=1,2,\cdots n).\end{aligned}\tag{16}$$

Equation (16) is the expressions of the three-order Lagrangian equations in a motional coordinate system. The second term on the right-hand side in Eq.(16) is the time rate of change of generalized force which comes from the coupling interaction between the velocity of mass centre $\dot{\boldsymbol{r}}_c^*$ and the acceleration $\ddot{\boldsymbol{r}}_0$ of the motional coordinate system S'. The third term on the right-hand side in Eq.(16) is the time rate of change of generalized force which comes from the coupling interaction of relative acceleration of particle and the translation acceleration of origin o of S' system with centripetal acceleration of rotation of the S' system. The fourth term on the right-hand side in Eq.(16) is the time rate of change of generalized force which comes from the coupling interaction of Coriolis acceleration with angular acceleration.

3. Discussion

In this section, based on the result of Eq.(16), we discuss several specific forms of the dynamical equation in relative motion.

First, if the S' system is also stationary, i.e. $\dot{\boldsymbol{r}}_0=\ddot{\boldsymbol{r}}_0=0$, and $\boldsymbol{\omega}=0$, then $\boldsymbol{R}_i=\boldsymbol{r}_i$ and Eq.(16) has the same form as Eq.(1).

Second, if the S' system has a translation motion but no rotation, i.e. $\ddot{\boldsymbol{r}}_0\neq0$, and $\boldsymbol{\omega}=0$, then Eq.(16) can be simplified as follows:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S_1}{\partial\ddot{q}_s}\right)-\frac{1}{2}\frac{\partial S_1}{\partial\dot{q}_s}\\ =&Q_s-\frac{1}{2}\sum_{i=1}^{N}m_i\ddot{\boldsymbol{r}}_0\cdot\left(2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\ddot{q}_s}-\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right),\\ &(s=1,2,\cdots n).\end{aligned}\tag{17}$$

Third, if the S' system has a rotation with one point fixed, then the origins of the S system and the S' system can be at the same point, i.e. $\boldsymbol{r}_0=\dot{\boldsymbol{r}}_0=\ddot{\boldsymbol{r}}_0=0$, and Eq.(16) will have the following form:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S_1}{\partial\ddot{q}_s}\right)-\frac{1}{2}\frac{\partial S_1}{\partial\dot{q}_s}\\ =&Q_s-\frac{1}{2}\sum_{i=1}^{N}m_i\ddot{\boldsymbol{r}}_0\cdot\left(2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\ddot{q}_s}-\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right)\\ &\cdot\left[\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_i)\right]\\ =&Q_s+\frac{1}{2}\sum_{i=1}^{N}m_i\left[\left(2\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\ddot{q}_s}-\frac{\partial\ddot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right)\cdot\boldsymbol{\omega}^2\boldsymbol{r}_{i\perp}\right.\\ &\left.+\left(\boldsymbol{\omega}\times\frac{\partial\dot{\boldsymbol{r}}_i^*}{\partial\dot{q}_s}\right)\cdot\left[4\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_i^*+2\dot{\boldsymbol{\omega}}^*\times\boldsymbol{r}_i\right]\right],\\ &(s=1,2,\cdots,n),\end{aligned}\tag{18}$$

where $\boldsymbol{r}_{i\perp}$ is the component of $\boldsymbol{r}_i$ perpendicular to the oz axis.

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