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First integrals of the discrete nonconservative and nonholonomic systems*

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In this paper we show that the first integrals of the discrete equation of motion for nonconservative and nonholonomic mechanical systems can be determined explicitly by investigating the invariance properties of the discrete Lagrangian. The result obtained is a discrete analogue of the generalized theorem of Noether in the Calculus of variations.

Keywords: discrete mechanics, nonconservative and nonholonomic mechanical systems, Noether's theorem, first integral

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1. Introduction

The symmetry problem of mechanical systems is one of the most important subjects in physics, and has been studied for a long time. Moreover, the symmetry of a mechanical system will be useful for integrating the equations of motion, since it is closely related to the invariants (or first integrals) of the system. Now, the invariants are used effectively for the numerical integration of various equations in mathematical physics. For example, for the equations of motion in classical mechanics, an attempt was made to devise a difference scheme in which the accuracy of approximate solutions can be raised higher by means of the invariants.[1] Furthermore, when a system of ordinary differential equations is numerically integrated over a long range, the invariants can be used to check the accuracy of solutions against the round-off errors.[2] Therefore, the approaches to find conserved quantities in continuous cases are extended to the discrete mechanical systems.

In 1970, Cadzow[3] motivated and discussed a one-freedom, discrete calculus of variations in which the fundamental problem is to select among all finite sequence $\{q_k\}, k = M-1, M, \cdots, N$, the one which extremizes the sum

$$J\{q_k\} = \sum_{k=M}^{N} F(k, q_k, q_{k-1}), \tag{1}$$

where $F(x, y, z)$ is a given function with continuous derivatives$F_y$andF_z. A necessary condition for J to be extremal is that $\{q_k\}$ satisfy a second-order difference equation

$$F_y(k, q_k, q_{k-1}) + F_z(k+1, q_{k+1}, q_k) = 0,$$
$$(k = M, \cdots, N-1), \tag{2}$$

where we have denoted

$$F_y(k, q_k, q_{k-1}) = \left.\frac{\partial F(x, y, z)}{\partial y}\right|_{x=k, y=q_k, z=q_{k-1}},$$

and

$$F_z(k+1, q_{k+1}, q_k) = \left.\frac{\partial F(x, y, z)}{\partial z}\right|_{x=k+1, y=q_{k+1}, z=q_k},$$

Equation (2) is called the discrete Euler–Lagrange equation.

In 1973 Logan[4] extended Noether's theorem to the discrete case and showed that the first integrals of Eq.(2) can be obtained by studying the invariance

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properties of $F(x,y,z)$, and further studied the multi-freedom and higher-order problems. In 1981 Maeda[5] constructed the constants of motion for the discrete time system in the case of continuous symmetries.

Consider that the above-mentioned theorems in constructing first integrals are all for the discrete unconstrained conservative mechanical systems. In 2001 Marsden and West[6] presented Noether's theorem for the discrete holonomic constraint systems. In this paper, we study the problem of constructing first integrals of discrete equations of motion for nonconservative and nonholonomic systems by investigating the invariance properties of the discrete Lagrangian. In Section 2, based on the discrete Lagrange–d'Alembert principle, we derive the discrete equations of motion for the nonconservative mechanical systems, present a new definition on invariance properties of the discrete Lagrangian and a theorem on first integrals of the discrete nonconservative systems, and we further discuss the first integrals of the discrete equations of motion for multi-freedom nonconservative systems. In Section 3, we give the discrete equations of motion for the mechanical systems with first-order linear nonholonomic constraints and a theorem related to first integrals. Finally, in Section 4, two simple examples are presented to illustrate the applications of the results.

2. The discrete nonconservative systems

2.1. The discrete equations of motion for nonconservative systems

In the continuous case, the integral Lagrange–d'Alembert principle for nonconservative systems is[7]

$$\delta\int L[t,q(t),\dot{q}(t)]\mathrm{d}t+\int Q''[t,q(t),\dot{q}(t)]\delta q\mathrm{d}t=0, \quad (3)$$

where $L=T-V$, $Q=Q'+Q''$, $Q'=-\dfrac{\partial V}{\partial q}$ are generalized conservative forces and Q'' are generalized nonconservative forces.

We define the discrete Lagrange–d'Alembert principle as

$$\sum_{k=M}^{N}\delta L_d(q_{k-1},q_k,k)+\sum_{k=M}^{N}Q''_d(q_{k-1},q_k,k)\delta q_k=0, \quad (4)$$

where $L_d(q_{k-1},q_k,k)$ is the discrete Lagrangian and $Q''_d(q_{k-1},q_k,k)$ is the discrete generalized nonconservative force.

Proposition 1 For nonconservative systems with Lagrangian $L(q,\dot{q},t)$ and generalized nonconservative force $Q''(q,\dot{q},t)$, the discrete Euler–Lagrange equations can be derived from principle (4):

$$\begin{aligned}&\frac{\partial L_d(q_{k-1},q_k,k)}{\partial q_k}+\frac{\partial L_d(q_k,q_{k+1},k+1)}{\partial q_k}\\&+Q''_d(q_{k-1},q_k,k)=0,\\&(k=M,\cdots,N-1).\end{aligned} \quad (5)$$

Proof From Eq.(4), one has

$$\begin{aligned}&\sum_{k=M}^{N}\left[\frac{\partial L_d(q_{k-1},q_k,k)}{\partial q_{k-1}}\delta q_{k-1}+\frac{\partial L_d(q_{k-1},q_k,k)}{\partial q_k}\delta q_k\right]\\&+\sum_{k=M}^{N}Q''_d(q_{k-1},q_k,k)\delta q_k\\=&\sum_{k=M}^{N-1}\left[\frac{\partial L_d(q_{k-1},q_k,k)}{\partial q_k}+\frac{\partial L_d(q_k,q_{k+1},k+1)}{\partial q_k}\right]\delta q_k\\&+\sum_{k=M}^{N-1}Q''_d(q_{k-1},q_k,k)\delta q_k=0,\end{aligned} \quad (6)$$

where we have used a discrete integration by parts (rearranging the summation) and the fact that $\delta q_{M-1}=\delta q_N=0$. If we now require that Eq.(6) be zero for any choice of δq_k, then we obtain the discrete Euler–Lagrange equation (5).

For convenience, Eq.(5) may be rewritten as follows:

$$\begin{aligned}&D_2L_d(q_{k-1},q_k,k)+D_1L_d(q_k,q_{k+1},k+1)\\&+Q''_d(q_{k-1},q_k,k)=0,\\&(k=M,\cdots,N-1),\end{aligned} \quad (7)$$

where

$$D_2L_d(q_{k-1},q_k,k)=\frac{\partial L_d(q_{k-1},q_k,k)}{\partial q_k},$$

and

$$D_1L_d(q_k,q_{k+1},k+1)=\frac{\partial L_d(q_k,q_{k+1},k+1)}{\partial q_k}.$$

2.2. First integrals of the discrete nonconservative systems

In the classical calculus of variations, Noether[8] in 1918 showed that if the variational integral is invariant under an infinitesimal transformation group, then a certain number of identities hold true. Under the additional assumption that the Euler–Lagrange equations for the system are satisfied, these identities reduce to expressions which are constant along the extremals. This is the well-known Noether theorem. In

1975 Djukić and Vujanović[9] generalized the Noether theorem to the mechanical systems with nonconservative forces.

For discrete nonconservative systems, similar conclusions can be drawn. In this section, it is shown that in a systematic procedure, the establishment of first integrals of the discrete Euler–Lagrange equation can be developed from a direct study of the invariance properties of the discrete Lagrangian $L_d(Rq_k, q_k, k)$, where R is the lag operator[10] defined as

$$R^r q_k = q_{k-r}. \tag{8}$$

Let us introduce the following infinitesimal transformation:

$$q_k^* = q_k + \varepsilon u_k, \quad (k = M, \cdots, N-1), \tag{9}$$

where ε is a parameter and $u_k = u(q_k, k)$ is a sequence depending upon k and q_k, $\quad k = M, \cdots, N-1$.

Definition 1 The discrete Lagrangian L_d $(Rq_k,$ $q_k,$ $k)$ is generalized-difference-invariant with respect to the infinitesimal transformation (9), if there exists a sequence $v(q_{k-1}, q_k, k)$, $k = M, \cdots, N-1$ such that

$$\begin{aligned}&\delta L_d[\varepsilon u(q_k, k)]\\ =&\varepsilon[\Delta v(q_{k-1}, q_k, k) - Q_d''(q_{k-1}, q_k, k)u(q_k, k)]\end{aligned} \tag{10}$$

for each k, where δL_d is given by the following formula:[4]

$$\begin{aligned}\delta L_d(\delta q_k) =&[D_2 L_d(q_{k-1}, q_k, k)\\ &+ D_1 L_d(q_k, q_{k+1}, k+1)]\delta q_k\\ &+ \Delta[-\delta q_{k-1} D_1 L_d(q_{k-1}, q_k, k)].\end{aligned} \tag{11}$$

Here Δ is the difference operator, $\Delta q_k = q_{k+1} - q_k$.

Based on the discrete Lagrange–d'Alembert principle, the following theorem concerning the first integral can be proven.

Theorem 1 If the discrete Lagrangian L_d $(Rq_k,$ $q_k,$ $k)$ is generalized-difference-invariant with respect to the infinitesimal transformation (9), and if Eq.(7) holds, then

$$\begin{aligned}&u(q_{k-1}, k-1)D_1 L_d(q_{k-1}, q_k, k)\\ &+ v(q_{k-1}, q_k, k) = \text{const},\\ &(k = M, \cdots, N-1).\end{aligned} \tag{12}$$

Proof From Eqs.(10) and (11), one has

$$\begin{aligned}&[D_2 L_d(q_{k-1}, q_k, k) + D_1 L_d(q_k, q_{k+1}, k+1)\\ &+ Q_d''(q_{k-1}, q_k, k)]\varepsilon u(q_k, k)\\ &+ \Delta[-\varepsilon u(q_{k-1}, k-1)D_1 L_d(q_{k-1}, q_k, k)]\\ =&\varepsilon \Delta v(q_{k-1}, q_k, k).\end{aligned}$$

Simplifying and using Eq.(7), one obtains

$$\begin{aligned}&\Delta[u(q_{k-1}, k-1)D_1 L_d(q_{k-1}, q_k, k)\\ &+ v(q_{k-1}, q_k, k)] = 0,\end{aligned}$$

whence Eq.(12) holds. This completes the proof.

Equation (12), which is a first-order difference relation, represents a first integral of the second-order discrete Euler–Lagrange equation given by Eq.(7).

2.3. The multi-freedom nonconservative systems

In this section, we discuss the first integrals of the discrete equation of motion for multi-freedom nonconservative systems. More precisely, the objective is to investigate the invariance properties of L_d $(Rq_k^1, \cdots,$ $Rq_k^n,$ $q_k^1, \cdots,$ $q_k^n,$ $k)$ with respect to the infinitesimal transformations

$$\begin{cases} q_k^{1*} = q_k^1 + \varepsilon u^1(q_k^1, q_k^2, \cdots, q_k^n, k),\\ \quad \cdots\cdots\\ q_k^{n*} = q_k^n + \varepsilon u^n(q_k^1, q_k^2, \cdots, q_k^n, k). \end{cases} \tag{13}$$

For simplicity, we denote

$$\bar{q}_k = \{q_k^1, q_k^2, \cdots, q_k^n\},$$

and

$$R\bar{q}_k = \{Rq_k^1, Rq_k^2, \cdots, Rq_k^n\}.$$

For a multi-freedom nonconservative system, proceeding in a fashion similar to the procedures in Sections 2.1 and 2.2, we define the discrete Lagrange–d'Alembert principle to be (in the following we will assume the summation over repeated indices on different levels as usual)

$$\begin{aligned}&\sum_{k=M}^{N} \delta L_d(\bar{q}_{k-1}, \bar{q}_k, k)\\ &+ \sum_{k=M}^{N} Q_{d,s}''(\bar{q}_{k-1}, \bar{q}_k, k)\delta q_{s,k} = 0,\\ &(s = 1, 2, \cdots, n).\end{aligned} \tag{14}$$

Based on the principle (14), we may derive the following discrete equations of motion:

$$\begin{aligned}&D_2 L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) + D_1 L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)\\ &+ Q_{d,s}''(\bar{q}_{k-1}, \bar{q}_k, k) = 0,\\ &(k = M, \cdots, N-1;\ s = 1, 2, \cdots, n).\end{aligned} \tag{15}$$

Definition 2 The discrete Lagrangian L_d ($R\bar{q}_k$, $\bar{q}_k$, k) is generalized difference invariant with respect to the infinitesimal transformations (13), if there exists a sequence $v(\bar{q}_{k-1}, \bar{q}_k, k)$, $k = M, \cdots, N-1$ such that

$$\begin{aligned}&\delta L_d[\varepsilon u^s(\bar{q}_k, k)]\\ =&\varepsilon[\Delta v(\bar{q}_{k-1}, \bar{q}_k, k) - Q''_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)u^s(\bar{q}_k, k)],\end{aligned} \quad (16)$$

for each k, where δL_d is given by the following formula[4]

$$\begin{aligned}\delta L_d(\delta\bar{q}_k) =&[D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\ &+ D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)]\delta q_{s,k}\\ &+ \Delta[-\delta q_{s,k-1}D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)],\\ &(k = M, \cdots, N-1; s = 1, 2, \cdots, n).\end{aligned} \quad (17)$$

The following theorem gives the conditions under which the explicit first integrals of Eq.(15) can be obtained.

Theorem 2 If L_d ($R\bar{q}_k$, $\bar{q}_k$, k) is generalized-difference-invariant with respect to the infinitesimal transformations (13), and if Eq.(15) holds true, then

$$\begin{aligned}&u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\ &+ v(\bar{q}_{k-1}, \bar{q}_k, k) = \text{const},\\ &(k = M, \cdots, N-1).\end{aligned} \quad (18)$$

Proof From Eqs.(16) and (17), one has

$$\begin{aligned}&[D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\ &+ D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)\\ &+ Q''_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)]\varepsilon u^s(\bar{q}_k, k)\\ &+ \Delta\Big[-\varepsilon u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\Big]\\ =&\varepsilon\Delta v(\bar{q}_{k-1}, \bar{q}_k, k).\end{aligned}$$

Simplifying and using Eq.(15), one obtains

$$\begin{aligned}&\Delta\Big[u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\ &+ v(\bar{q}_{k-1}, \bar{q}_k, k)\Big] = 0,\end{aligned}$$

whence Eq.(18) holds.

3. The discrete nonholonomic systems

3.1. The discrete equations of motion for the first-order linear nonholonomic systems

In this section, we propose a discrete version of the Lagrange–d'Alembert principle for nonholonomic systems. Before doing so, we first recall the general picture in the continuous case.

We consider a mechanical system whose configuration is determined by the n generalized coordinates $q_s (s = 1, 2, \cdots, n)$. Let its motion be subjected to g ideal first-order linear nonholonomic constraints

$$\begin{aligned}&\omega_{\beta,s}(t, q_1, q_2, \cdots, q_n)\dot{q}_s = 0,\\ &(\beta = 1, 2, \cdots, g;\ s = 1, 2, \cdots, n).\end{aligned} \quad (19)$$

The equations of motion can be written in the form[11]

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} - \lambda_\beta\omega_{\beta,s} = 0,\\ &(\ \beta = 1, 2, \cdots, g;\ s = 1, 2, \cdots, n),\end{aligned} \quad (20)$$

where L is the Lagrangian and λ_β are the constraint multipliers.

Equation (20) can be considered as a holonomic system under certain conditions, where Eq.(19) is regarded as the particular first integrals of Eq.(20). The constraint multipliers λ_β can be expressed as explicit functions of the variables t, $\bar{q}$, $\dot{\bar{q}}$ before integrating the differential equations of motion.[11] Thus Eq.(20) can be expressed as

$$\begin{aligned}&\frac{d}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s} - \frac{\partial L}{\partial q_s} - \Lambda_s = 0,\\ &(s = 1, 2, \cdots, n),\end{aligned} \quad (21)$$

where

$$\begin{aligned}&\Lambda_s = \Lambda_s(\bar{q}, \dot{\bar{q}}, t) = \lambda_\beta(t, \bar{q}, \dot{\bar{q}})\omega_{\beta,s}(t, \bar{q}),\\ &(\beta = 1, 2, \cdots, g;\ s = 1, 2, \cdots, n).\end{aligned} \quad (22)$$

Equation (21) is called the equation of motion of the holonomic system corresponding to the nonholonomic system (19) and (20).

Now, we turn to the discrete version of nonholonomic mechanics. Let Q be an n-dimensional configuration manifold and $L_d : Q \times Q \to \mathbf{R}$ a smooth map playing the role of a discrete Lagrangian. The action sum is the map $J : Q^{N+1} \to \mathbf{R}$ defined by

$$J = \sum_{k=M}^{N} L_d(\bar{q}_{k-1}, \bar{q}_k, k), \quad (23)$$

where $q_k \in Q$ for $k \in \{M-1, M, \cdots, N\}$ and k is the discrete time. In the unconstrained case, the discrete variational principal states that the evolution equations extremize the action sum with respect to all possible sequences of $N - M - 1$ points, given the

fixed end points q_{M-1}, q_N. This means that at each point $q \in Q$, the allowed variations are given in the whole tangent space T_qQ. However, in the nonholonomic case, we must restrict the allowed variations. These are given exactly by the distribution Ω. In addition, we will consider a discrete constraint space $\Omega_d \subset Q \times Q$ with the same dimension as Ω such that $(q, q) \in \Omega_d$ for all $q \in Q$. This discrete constraint space imposes constraints on the solution sequence $\{q_k\}$, namely $(q_k, q_{k+1}) \in \Omega_d$.

Consequently, to develop the discrete nonholonomic mechanics, one needs three ingredients: a discrete Lagrangian L_d, a constraint distribution Ω on Q and a discrete constraint space Ω_d. Note that the discrete mechanics can also be seen within this framework, where $\Omega = TQ$ and $\Omega_d = Q \times Q$.

Then, we define the discrete Lagrange–d'Alembert principle to be the extremization of Eq.(23) among the sequence of points $\{q_k\}$ with given fixed end points q_{M-1} and q_N, where the variations must satisfy $\delta q_k \in \Omega_{q_k}$ and $(q_k, q_{k+1}) \in \Omega_q$ for all $k \in \{M, M+1, \cdots, N-1\}$. This leads to the set of equations

$$\begin{aligned}&[D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\&+ D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)]\delta q_{s,k} = 0,\\&(s = 1, 2, \cdots, n;\ k = M, M+1, \cdots, N-1),\end{aligned} \tag{24}$$

where $\delta q_k \in \Omega_{q_k}$, along with $(q_k, q_{k+1}) \in \Omega_d$. If $\omega_d^\beta : Q \times Q \to \mathbf{R}, \beta \in \{1, 2, \cdots, g\}$, are functions whose annihilation defines Ω_d, what we have obtained is the following discrete equation of motion for nonholonomic systems:[12]

$$\begin{aligned}&D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) + D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)\\&+ \lambda_\beta(\bar{q}_{k-1}, \bar{q}_k, k)\omega_{\beta,s}(\bar{q}_k) = 0,\end{aligned} \tag{25a}$$

$$\omega_\beta(\bar{q}_{k-1}, \bar{q}_k, k) = 0, \tag{25b}$$

where $\lambda_\beta, \beta \in \{1, 2, \cdots, g\}$, is a set of Lagrange multipliers, The third term in the left-hand side of Eq.(25a) represents the discrete force of constraint.

Proceeding in a fashion similar to the continuous case, the constraint multipliers λ_β can be expressed as explicit functions of the variables k, $\bar{q}_{k-1}$, $\bar{q}_k$ before solving the difference equations (25a) and (25b). Thus Eq.(25a) can be expressed as

$$\begin{aligned}&D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) + D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)\\&+ \Lambda_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) = 0,\end{aligned} \tag{26}$$

where $\Lambda_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) = \lambda_\beta(\bar{q}_{k-1}, \bar{q}_k, k)\omega_s^\beta(\bar{q}_k)$, and then Eq.(26) is called the equation of motion of the discrete holonomic system corresponding to the discrete nonholonomic system (25a) and (25b).

3.2. First integral of the discrete nonholonomic systems

For the discrete nonholonomic systems, a similar procedure as in Section 2.2 can be described as follows.

Definition 3 The discrete Lagrangian L_d $(R\bar{q}_k, \bar{q}_k, k)$ is generalized-difference-invariant with respect to the infinitesimal transformation (13), if there exists a sequence $v(\bar{q}_{k-1}, \bar{q}_k, k)$, $k = M, \cdots, N-1$ such that

$$\begin{aligned}&\delta L_d[\varepsilon u^s(\bar{q}_k, k)]\\=&\varepsilon[\Delta v(\bar{q}_{k-1}, \bar{q}_k, k) - \Lambda_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)u^s(\bar{q}_k, k)],\\&(s = 1, 2, \cdots, n),\end{aligned} \tag{27}$$

for each k, where δL_d is given by Eq.(17)

In terms of Eqs.(26) and (27), the following theorem related to the first integral can be proven.

Theorem 3 The discrete holonomic system (26) corresponding to the discrete nonholonomic system (25(a)) and (25(b)) possesses the following first integrals:

$$\begin{aligned}&u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\&+ v(\bar{q}_{k-1}, \bar{q}_k, k) = \text{const.}\\&(k = M, \cdots, N-1;\ s = 1, 2, \cdots, n),\end{aligned} \tag{28}$$

if the discrete Lagrangian L_d $(R\bar{q}_k, \bar{q}_k, k)$ is generalized-difference-invariant with respect to the infinitesimal transformations (13), namely $\delta u^s(\bar{q}_k, k)$ admits Eq.(27).

Proof From Eqs.(17) and (27), one has

$$\begin{aligned}&[D_2L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k) + D_1L_{d,s}(\bar{q}_k, \bar{q}_{k+1}, k+1)\\&+ \Lambda_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)]\varepsilon u^s(\bar{q}_k, k)\\&+ \Delta[-\varepsilon u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)]\\=&\varepsilon\Delta v(\bar{q}_{k-1}, \bar{q}_k, k).\end{aligned}$$

Simplifying and using Eq.(26), one obtains

$$\begin{aligned}&\Delta[u^s(\bar{q}_{k-1}, k-1)D_1L_{d,s}(\bar{q}_{k-1}, \bar{q}_k, k)\\&+ v(\bar{q}_{k-1}, \bar{q}_k, k)] = 0,\end{aligned}$$

whence Eq.(28) holds.

4. Examples

To illustrate the various aspects of the theory developed in the preceding sections, we now give two examples.

Example 1 Consider a nonconservative system with discrete Lagrangian

$$L_d = \frac{1}{2} m_n (q_k - q_{k-1})^2, \tag{29}$$

and discrete nonconservative force

$$Q''_d = q_k - q_{k-1}, \tag{30}$$

where $m_1, \cdots, m_N$ are given constants. Then the action sum is

$$J = \sum_{k=1}^{N} \frac{1}{2} m_k (q_k - q_{k-1})^2. \tag{31}$$

From Eq.(7), the discrete Euler-Lagrange equation is a second-order difference equation

$$m_k(q_k - q_{k-1}) - m_{k+1}(q_{k+1} - q_k) + (q_k - q_{k-1}) = 0. \tag{32}$$

Direct verification shows that

$$L_d(Rq_k, q_k, k) = \frac{1}{2} m_k (q_k - Rq_k)^2$$

is generalized-difference-invariant (with $v = q_{k-1}$) with respect to the infinitesimal transformation

$$q^*_k = q_k + \varepsilon. \tag{33}$$

Hence, by Theorem 1, we immediately obtain a first integral

$$D_1 L_d(q_{k-1}, q_k, k) + v(q_{k-1}, q_k, k) = \text{const},$$

or

$$-m_k(q_k - q_{k-1}) + q_{k-1} = \text{const}. \tag{34}$$

Example 2 Consider a nonholonomic system with discrete Lagrangian

$$L_d = \frac{1}{2}[(q_{1,k} - q_{1,k-1})^2 + (q_{2,k} - q_{2,k-1})^2 + (q_{3,k} - q_{3,k-1})^2], \tag{35}$$

and with its motion subjected to a nonholonomic constraint

$$\omega = (q_{2,k} - q_{2,k-1}) - q_{3,k-1}(q_{1,k} - q_{1,k-1}) = 0. \tag{36}$$

From Eq.(26), one has

$$(q_{1,k} - q_{1,k-1}) - (q_{1,k+1} - q_{1,k}) - \lambda q_{3,k-1} = 0, \tag{37a}$$

$$(q_{2,k} - q_{2,k-1}) - (q_{2,k+1} - q_{2,k}) + \lambda = 0, \tag{37b}$$

$$(q_{3,k} - q_{3,k-1}) - (q_{3,k+1} - q_{3,k}) = 0. \tag{37c}$$

Using Eqs.(36), (37a), (37b) and (37c), we have

$$\lambda = \frac{(q_{1,k} - q_{1,k-1})(q_{3,k} - q_{3,k-1})}{1 + q_{3,k-1} q_{3,k}}. \tag{38}$$

Direct verification shows that

$$L_d(\bar{q}_{k-1}, \bar{q}_k, k) = \frac{1}{2}[(q_{1,k} - q_{1,k-1})^2 + (q_{2,k} - q_{2,k-1})^2 + (q_{3,k} - q_{3,k-1})^2] \tag{39}$$

is generalized-difference-invariant (with $v = 0$) with respect to the following infinitesimal transformations:

$$q^*_{1,k} = q_{1,k}, \tag{40a}$$

$$q^*_{2,k} = q_{2,k}, \tag{40b}$$

$$q^*_{3,k} = q_{3,k} + \varepsilon. \tag{40c}$$

Therefore, by Theorem 3, we have

$$D_1 L_{d,3}(\bar{q}_{k-1}, \bar{q}_k, k) = \text{const},$$

or

$$-(q_{3,k} - q_{3,k-1}) = \text{const}. \tag{41}$$

Remark Due to the discrete nature of the problem in the variable k, it is not possible to vary k continuously, and consequently we have not obtained the " energy" integral in the classical, continuous calculus of variations, but only the "momentum" integral. However, we desire to obtain the "energy" integral for discrete mechanical systems, so we need to extend the framework to time variations in addition to the usual configuration variable variations. This problem will be discussed in the future.

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