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Generalized reduced-order synchronization of chaotic system based on fast slide mode*

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A new kind of generalized reduced-order synchronization of different chaotic systems is proposed in this paper. It is shown that dynamical evolution of third-order oscillator can be synchronized with the canonical projection of a fourth-order chaotic system generated through nonsingular states transformation from a cell neural net chaotic system. In this sense, it is said that generalized synchronization is achieved in reduced-order. The synchronization discussed here expands the scope of reduced-order synchronization studied in relevant literatures. In this way, we can achieve generalized reduced-order synchronization between many famous chaotic systems such as the second-order Düffing system and the third-order Lorenz system by designing a fast slide mode controller. Simulation results are provided to verify the operation of the designed synchronization.

Keywords: chaos generalized reduced-order synchronization fast slide controller
PACC: 0545

1. Introduction

In the last decade, synchronization of chaos has become a field of great interest, and several types of synchronization, such as complete synchronization (CS),[1] generalized synchronization (GS),[2] lag synchronization (LS)[3] and phase synchronization (PS),[4] have been widely studied. Actually, the above problem of synchronization is to design a force for coupling two chaotic systems.[5,6]

CS was the first discovered and is the simplest form of synchronization in chaotic systems. It consists in a perfect hooking of the chaotic trajectories of two systems, which is achieved by means of a coupling signal, in such a way that they remain in step with each other in the course of the time. This mechanism was first shown to occur when two identical chaotic systems are coupled unidirectionally, provided that the conditional Lyapunov exponents of the subsystem to be synchronized are all negative.[1]

Coupled nonidentical oscillatory or rotatory systems can reach an intermediate regime (PS), wherein a locking of the phases is produced, while correlation in the amplitudes remains weak.[7] The transition to PS for two coupled oscillators has been firstly characterized with reference to the Rössler system.[7]

GS goes further in using completely different systems and associating the output of one system to a given function of the output of the other system,[6] and the GS can be observed in real experimental systems.[8,9]

LS is a step between PS and CS. It implies the asymptotic boundedness of the difference between the output of one system at time t and the output of the other shifted in time of a time lag.[3] This implies that the two outputs have a lock in their phases and amplitudes, but with a time lag.[3]

Nevertheless, in many real systems, the synchronization is carried out even though the oscillators have different orders, such as thalamic neurons, as it is reasonable if their order is different from the one of the hippocampal neurons. Another example is the synchronization between heart and lung, but this kind of synchronization between chaotic systems with different orders has received less attention.

This paper address a kind of problem of the synchronization of chaotic systems with different orders, that is, two strictly different chaotic systems can

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achieve synchronization by designing a force; the synchronization is not the partial synchronization, as the partial synchronization is for coupling two chaotic systems which have an equal order. It is the problem of synchronizing a slave system with the projection of a master system, that is to say, all the states of the slave system are synchronous in some sense with the states of the master system.[13]

Femat and Solís-Perales[13] proposed a method to synchronize the Chua oscillator and Düffing equation based on time derivative of system output along the master/slave vector fields. Wang and Ge[14] used an adaptive backstepping design to synchronize the Rössler system and Düffing equation. Bowong[15] designed an exponential controller to show that Düffing equation can be synchronized with the canonical projection of a fourth-order chaotic system. These techniques show that it is feasible for synchronization between different orders of chaotic systems even if their parameters are uncertain or time-varying.

This paper address a class of synchronization of chaotic systems with different orders, the slave system is described in the following form:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \quad \vdots \\ \dot{x}_{m-1} = x_m, \\ \dot{x}_m = f(\theta, x_1, x_2, \cdots, x_m) + u, \end{cases} \tag{1}$$

where $x_i, i = 1, 2, \cdots, m$, are the states of system (1), θ stands for the definite or uncertain parameters, u is the coupling force.

The master system is assumed to have the form of chaotic systems such as $\dot{X} = F(X)$, where $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, but it has the order higher than that of the slave system, that is to say, $n > m$; moreover, it can be transformed into the following form related to the slave chaotic system through a nonsingular states transformation:

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_3, \\ \quad \vdots \\ \dot{y}_{m-1} = y_m, \\ \dot{y}_m = f_1(\theta, y_1, y_2, \cdots, y_n), \\ \quad \vdots \\ \dot{y}_n = f_{n-m+1}(y_1, y_2, \cdots, y_n). \end{cases} \tag{2}$$

We call system (2) the resembling chaotic system of the original system, because system (2) does not change the properties of the original system. Obviously, system (2) has an m-order canonical projection. The problem of synchronization for systems with different order, which will be discussed, is the synchronization of system (1) and the canonical projection of system (2), and we call this kind of reduced order synchronization as reduced-order resemblance synchronization between the slave system and the original master system. The reduced-order synchronization discussed in Ref.[15] is a case of what we discuss here.

In fact, the condition of the master system is not strict, as many systems can be converted into form (2), for example, the following Chen system[10]

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - xz + cy, \\ \dot{z} = xy - bz, \end{cases} \tag{3}$$

can be converted into form (2) by states transformation

$$Z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^{\mathrm{T}} = P \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}},$$

where

$$P = \begin{bmatrix} 1/a & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a nonsingular matrix, and we have

$$\begin{cases} \dot{z}_1 = z_2, \\ \dot{z}_2 = (2ca - a^2)z_1 + (c - a)z_2 - az_1z_3, \\ \dot{z}_3 = az_1(az_1 + z_2) - bz_3. \end{cases} \tag{4}$$

In the same way, the following cell neural network chaotic system[11]

$$\begin{cases} \dot{y}_1 = y_3 - y_4, \\ \dot{y}_2 = 2y_2 + y_3, \\ \dot{y}_3 = 14y_1 - 14y_2, \\ \dot{y}_4 = 100y_1 - 100y_4 + 100(|y_4 + 1| - |y_4 - 1|), \end{cases} \tag{5}$$

can also be transformed into form (2) through states transformation

$$\begin{aligned} Z &= \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \end{bmatrix} \\ &= P \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}, \end{aligned} \tag{6}$$

where

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 14 & -10 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a nonsingular matrix, then system (5) becomes the following form:

$$\begin{cases} \dot{m}_1 = m_2, \\ \dot{m}_2 = m_3, \\ \dot{m}_3 = 28m_1 - 28m_2 + 2m_3 - 14m_4, \\ \dot{m}_4 = \dfrac{50}{7}(14m_1 - 2m_2 + m_3) - 100m_4 \\ \qquad +100(|m_4 + 1| - |m_4 - 1|). \end{cases} \tag{7}$$

The organization of the paper is as follows. In section 2, the formulation of the problem is presented. Section 3 deals with the problem of reduced-order synchronization via direct feedback and adaptive feedback synchronization. In section 4, some simulations are carried out to illustrate the method. Conclusions come in the last section.

2. Description of the problem

For the synchronization of strictly different chaotic systems with the forms (1) and (2), we will consider two different situations: one is that the system (1) is definite, that is the parameter θ is known; then let $e_i = x_i - y_i, i = 1, 2, \cdots, m$, and we have the error equations:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \quad \vdots \\ \dot{e}_m = f(\theta, x_1, x_2, \cdots, x_m) \\ \qquad - f_1(y_1, y_2, \cdots, y_n) + u. \end{cases} \tag{8}$$

The other one is that θ in $f(\theta, x_1, x_2, \cdots, x_m)$ is uncertain. Our objective here is to design u so that the synchronization between systems (1) and (2) can be achieved in both situations. Thus the slave chaotic system becomes synchronous with the canonical projection of the master system (2).

Without loss of the generality, we will use an electronic chaotic oscillator of canonical structure[12] as a slave system; the dynamics of this model are given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -a[x_3 + x_2 + x_1 - f(x_1)], \end{cases} \tag{9}$$

where a is the control parameter for this oscillator, $f(x_1) = \mathrm{sgn}(x_1)$ is the model nonlinearity. This oscillator is known to give a double-scroll-like chaotic attractor in the range $0.49 < a < 1$. Figure 1 shows the phase portrait of system (9). System (7) is used as a master chaotic system. Figure 2 shows the phase portrait of system (7).

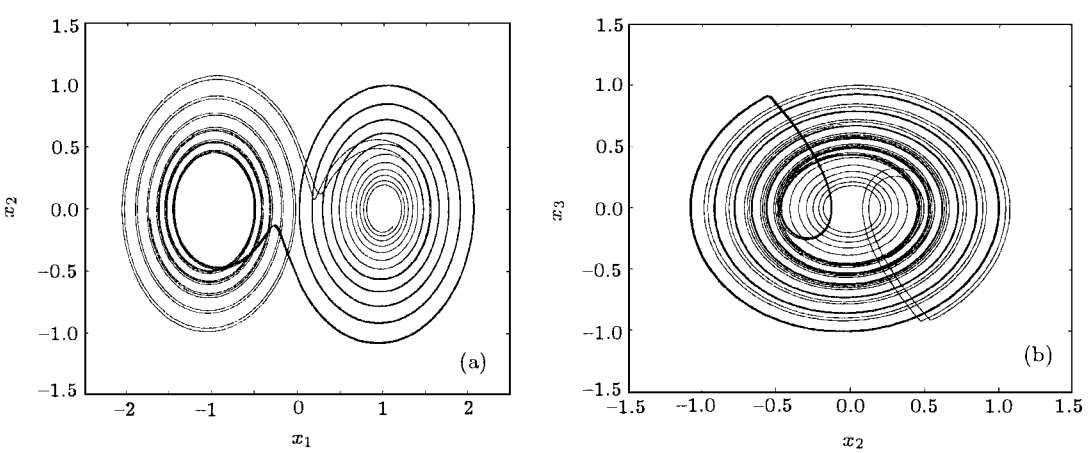


Fig.1. Phase portraits of chaotic system (9). (a) Phase portrait of $x_1 - x_2$; (b) Phase portrait of $x_2 - x_3$.

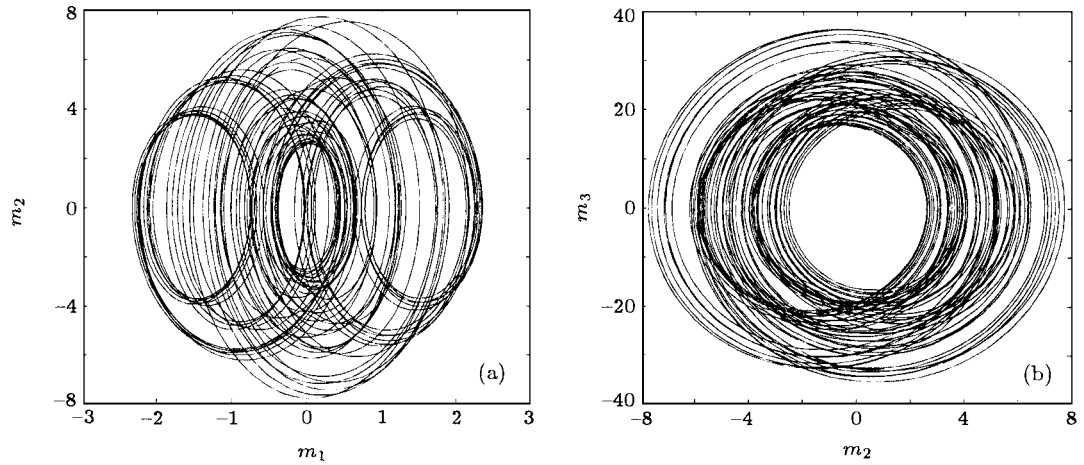


Fig.2. Phase portraits of chaotic system (7). (a) Phase portrait of $m_1 - m_2$; (b) Phase portrait of $m_2 - m_3$.

Obviously, there is a distinct difference between attractors of systems (7) and (9); both systems (7) and (9) are not synchronized in phase. Here we will give a projection of system (7) in a canonical plane, i.e., to lead the attractor of Eq.(9) to the geometrical properties of a projection of system (7).

3. Direct feedback synchronization

For chaotic systems (1) and (2), first, let us consider the situation that parameters of the slave chaotic system is definite, the difference can be described by $e_i = x_i - m_i, i = 1, 2, \ldots, m$; then the error dynamic system can be described as follows:

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \quad \vdots \\ \dot{e}_m = f(\theta, x_1, x_2, \cdots, x_m) - f_1(y_1, y_2, \cdots, y_n) + u. \end{cases} \tag{10}$$

If we choose

$$\begin{aligned} u =& f(\theta, x_1, x_2, \cdots, x_m) - f_1(y_1, y_2, \cdots, y_n) \\ &+ k_1 e_1 + k_2 e_2 + k_3 e_3 + \cdots + k_m e_m, \end{aligned} \tag{11}$$

it leads to

$$\begin{cases} \dot{e}_1 = e_2, \\ \dot{e}_2 = e_3, \\ \quad \vdots \\ \dot{e}_m = k_1 e_1 + k_2 e_2 + \cdots + k_m e_m, \end{cases} \tag{12}$$

where $k_1, k_2, \cdots, k_m$ are constant parameters of the control, which are to be computed by the following method. System (12) can be written in the form: $\dot{e} = Ae$, where matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ k_1 & k_2 & k_3 & \cdots & k_m \end{bmatrix}. \tag{13}$$

Then $k_1, k_2, \cdots, k_m$ are chosen such that A has all its eigenvalues at the open left-hand complex plane, that is $K = (k_1, k_2, \cdots, k_m)^{\mathrm{T}}$ should be a Hurwitz vector, with all the roots of the polynomial

$$p(s) = s^m + k_m s^{m-1} + k_{m-1} s^{m-2} + \cdots + k_1$$

having negative real parts, then the error $e = (e_1, e_2, \cdots, e_m)^{\mathrm{T}}$ is stable at the origin, i.e. system (9) becomes synchronized with the canonical projection of the master system (7).

Nevertheless, the controller (11) is required to have the parameter of the slave system: if the parameter is not certain or it is time-varying, then controller (11) is not appropriate. In next section, we give a robust fast slide mode controller to achieve the reduced-order synchronization of the slave system, i.e., all the states of the slave system can be synchronized, in some sense, with the states of the master system.

4. Robust fast slide mode synchronization

Before introducing the constructed controller, we will construct the sliding surface, and first give a lemma described as follows:

Lemma 1[16] The fast sliding mode is given by

the following equation

$$s = \dot{x} + \alpha x + \beta x^{q/p}, \tag{14}$$

where $x \in R$, α, β are positive constants, q, p are positive odd constants, then for any initial value of state $x(t)$ at $t = 0$, i.e., $x(0)$, the solution of Eq.(14) will reach $x = 0$ in a finite time t_s determined by

$$t_s = \frac{p}{\alpha(p-q)} \ln \frac{\alpha x(0)^{(p-q)/p} + \beta}{\beta}. \tag{15}$$

Now, we consider the following sliding surface

$$\begin{aligned} s_0 =& e_1(t), \\ s_1 =& \dot{s}_0 + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0}, \\ s_2 =& \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{q_1/p_1}, \\ & \vdots \\ s_{n-1} =& \dot{s}_{n-2} + \alpha_{n-2} s_{n-2} + \beta_{n-2} s_{n-2}^{q_{n-2}/p_{n-2}}. \end{aligned} \tag{16}$$

where $\alpha_i, \beta_i, i = 0, 1, \ldots, n-2$ are positive constants, $p_i, q_i, i = 0, 1, \ldots, n-2$ are positive odd constants, and $p_i > q_i$; from Eq.(14), we can obtain $s_{i+1} = \dot{s}_i + \alpha_i s_i + \beta_i s_i^{q_i/p_i}$, s_i will reach zero in a finite time when s_{i+1} reaches zero by lemma 1,and so it can be concluded that s_0 reaches zero in a finite time.

For the error dynamic system (10), as a is uncertain and the states of the chaotic systems are bounded, we assume that

$$f(\theta, x_1, x_2, \cdots, x_m) \le M, \tag{17}$$

where M is a positive constant. To achieve the condition indicated in Lemma 1, a scheme is proposed as follows:

$$u(t) = -\gamma\eta \mathrm{sign}(s_{n-1}(t)), \quad \gamma > 1 \tag{18}$$

where

$$\begin{aligned} \eta =& M + |f_1(y_1, y_2, \cdots, y_n)| \\ &+ \sum_{j=0}^{n-2} \alpha_j \left| s_j^{(n-1)} \right| + \sum_{j=0}^{n-2} \beta_j \left| \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j} \right|. \end{aligned} \tag{19}$$

So we have the following theorem.

Theorem 1 For the error dynamic system (10), the motion of the sliding mode (16) is asymptotically stable, if the control $u(t)$ is given by Eq.(18).

Proof From Eq.(16), we can obtain

$$\dot{s}_{n-1} = s_0^{(n)} + \sum_{j=0}^{n-2} \alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2} \beta_j \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j}; \tag{20}$$

from Eq.(10), we have

$$\begin{aligned} \dot{s}_{n-1} =& M + |f_1(y_1, y_2, \cdots, y_n)| + u(t) \\ &+ \sum_{j=0}^{n-2} \alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2} \beta_j \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j}. \end{aligned} \tag{21}$$

From Eq.(20), we have

$$\begin{aligned} & s_{n-1}(t)\dot{s}_{n-1}(t) \\ =& s_{n-1}(t) \Bigg(u(t) + M + |f_1(y_1, y_2, \cdots, y_n)| \\ &+ \sum_{j=0}^{n-2} \alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2} \beta_j \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j} \Bigg). \end{aligned}$$

Let $F(X) = M + |f_1(y_1, y_2, \cdots, y_n)|$, then

$$\begin{aligned} & s_{n-1}(t)\dot{s}_{n-1}(t) \\ \le& |s_{n-1}(t)| \left(|F(X)| + \sum_{j=0}^{n-2} \alpha_j \left| s_j^{(n-1)} \right| \right. \\ &\left. + \sum_{j=0}^{n-2} \beta_j \left| \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j} \right| \right) + u(t) s_{n-1}(t), \end{aligned} \tag{22}$$

$$\begin{aligned} u^2(t) =& (-\gamma\eta \mathrm{sign}(s_{n-1}(t)))(-\gamma\eta \mathrm{sign}(s_{n-1}(t))) \\ =& \gamma^2\eta^2 (\mathrm{sign}\,(s_{n-1}(t))^2. \end{aligned} \tag{23}$$

Since $s_{n-1}^2(t) \ge 0$, we have

$$\begin{aligned} & -\gamma\eta \mathrm{sign}\,(s_{n-1}(t)) u(t) s_{n-1}^2(t) \\ =& \gamma^2\eta^2 (\mathrm{sign}\,(s_{n-1}(t))^2 s_{n-1}^2(t), \end{aligned} \tag{24}$$

$$-\gamma\eta s_{n-1}(t) |s_{n-1}(t)| u(t) = \gamma^2\eta^2 s_{n-1}^2(t),$$

$$s_{n-1}(t)u(t) = -\gamma\eta |s_{n-1}((t)|. \tag{25}$$

From Eq.(18), and replacing Eq.(22) with Eq.(25), we obtain

$$s_{n-1}(t)\dot{s}_{n-1}(t) \le |s_{n-1}(t)| \eta(1-\gamma). \tag{26}$$

Since $\gamma > 1$ has been selected in Eq.(26), so $s_{n-1}(t)\dot{s}_{n-1}(t) < 0$ is always satisfied. Thus the proof is completed, and the error dynamic system (10) is asymptotically stable at the origin.

5. Control simulation and results

Example 1 The slave system is a chaotic Düffing–Holmes oscillator described by

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -p_1 x_1(t) - p_2 x_2(t) \\ \qquad\quad -x_1^3(t) + q\cos(w_1 t) + u, \end{cases} \tag{27}$$

with parameter values of $p_1 = -1$, $p_2 = 0.25$, $q = 0.3$ and $w_1 = 1.0$. This system displays a chaotic

behaviour as shown in Fig.3, here we assume that p_1, p_2, q, w_1 all are uncertain. The master system is Eq.(4) converted through a nonsingular states transformation from Chen system. The phase portrait is depicted in Fig.4. From theorem 1 for $M = 5$, $\gamma = 1.8$, the controller can be described by

$$\begin{cases} s_1 = e_2 + e_1 + e_1^{3/5}, \\ \eta = M + \left|(2ca - a^2)z_1 + (c - a)z_2 - az_1z_3\right| \\ \qquad + |e_2| + \dfrac{3}{5}\left|e_1^{-2/5}e_2\right|, \\ u(t) = -\gamma\eta \mathrm{sign}\,(s_1(t)). \end{cases} \tag{28}$$

The phase portrait of Eq.(27) under the condition of $u(t)$ is shown in Fig.5. One can see that Fig.5 is a mirror reflection of Fig.4. The time evolutions of states are shown in Fig.6.

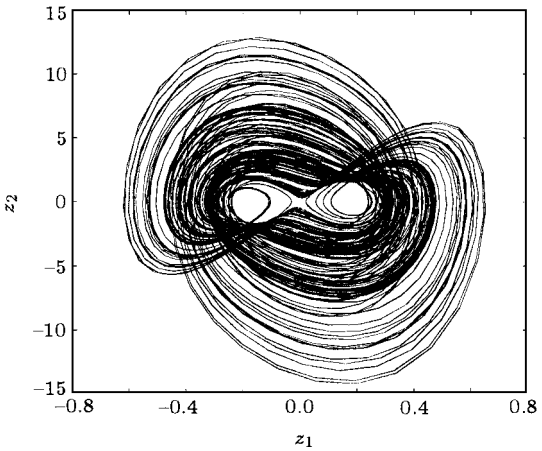


Fig.4. Phase portrait of system (4).

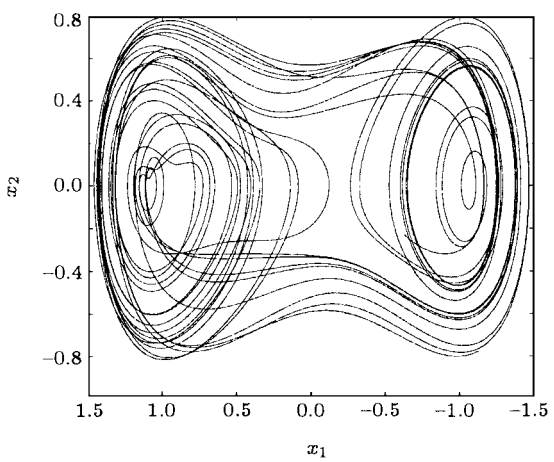


Fig.3. Phase portrait of system (27).

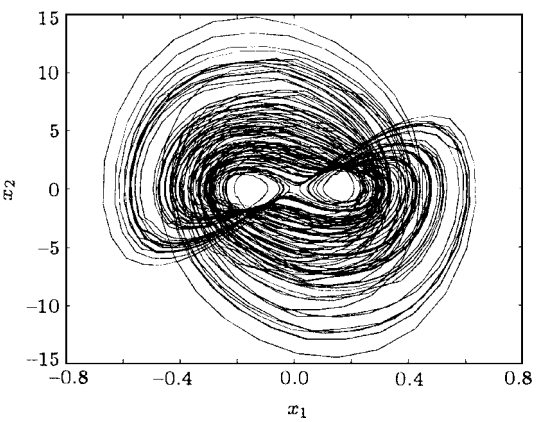


Fig.5. Phase portrait of system (27) under the control of (28).

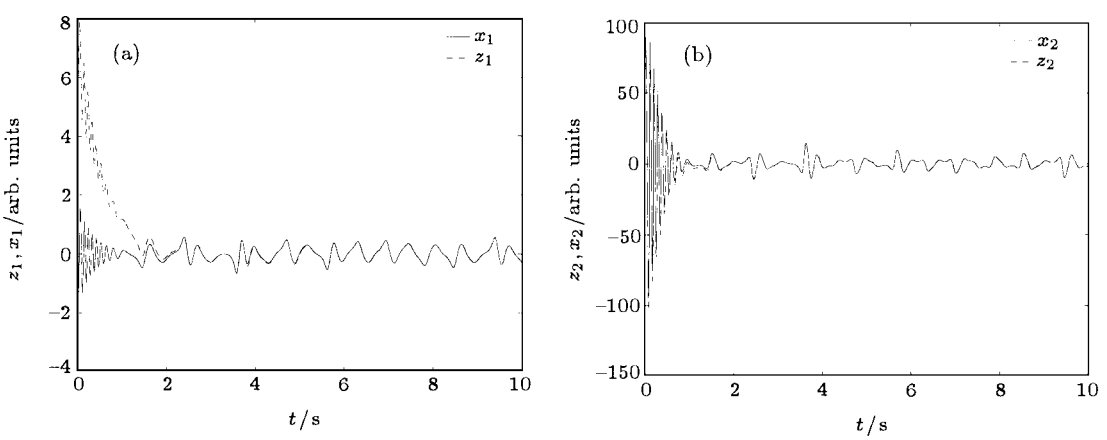


Fig.6. Synchronization of the master and slave systems via control. (a) $x_1(-)$ and $m_1(\cdots)$; (b) $x_2(-)$ and $m_2(\cdots)$.

Example 2 The slave system is an electronic chaotic oscillator described by Eq.(9). Here it is assumed that α is uncertain. The master system is Eq.(7) converted through a nonsingular states transformation from cell neural network chaotic system (5). From theorem 1, for $M = 25$, $\gamma = 1.8$, the controller can be described by Eq.(18). Figure 7 shows the synchronization of states between the master and slave systems.

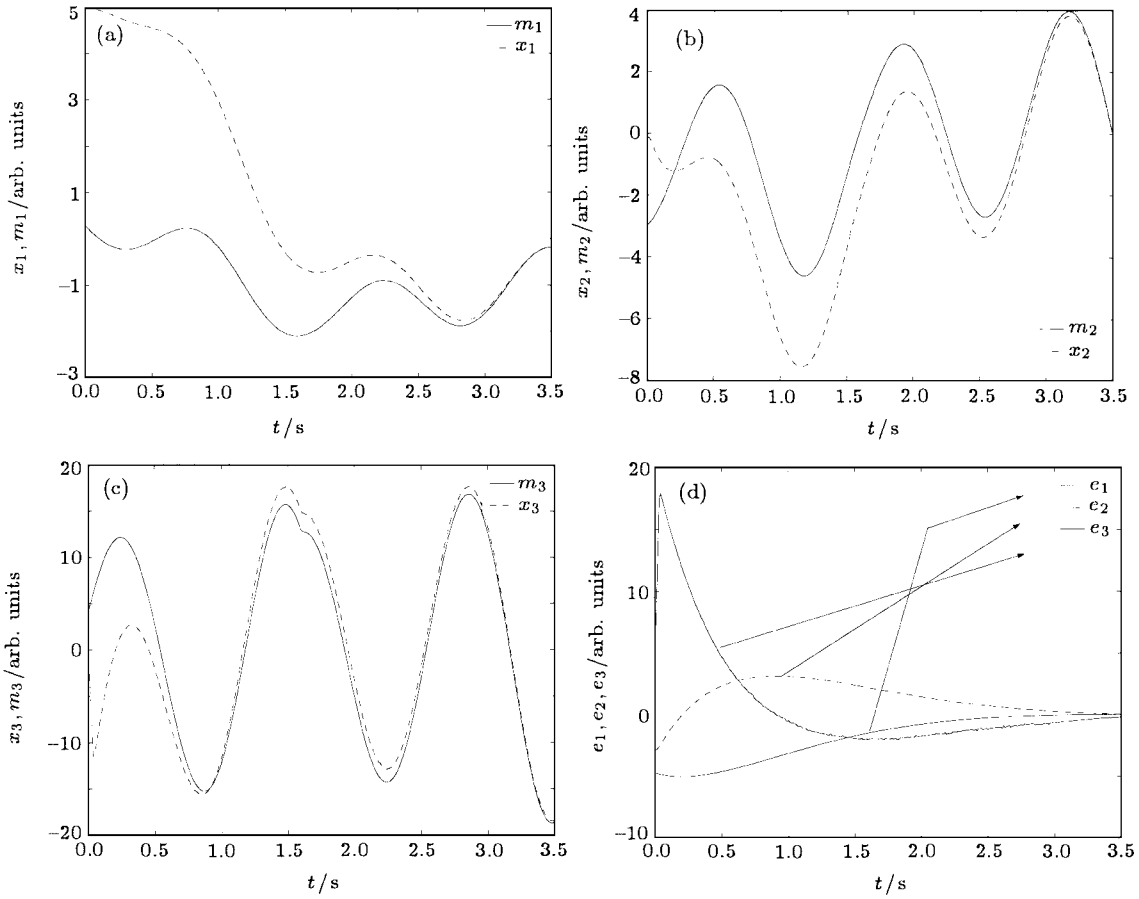


Fig.7. Evolution of synchronization error of the master and slave systems. (a) $m_1(-)$ and $x_1(\cdots)$; (b) $m_2(-)$ and $x_2(\cdots)$; (c) $m_3(-)$ and $x_3(\cdots)$; (d) e_1, e_2 and e_3.

6. Conclusion

In this paper, we present a kind of new generalized synchronization between different-order chaotic systems, i.e. the generalized reduced-order synchronization, and a fast slide mode controller is given to control the slave system to become a canonical projection of the master system. In this sense, many different-order chaotic systems can fulfill reduced-order synchronization as long as the master system can be converted into a class of format through a nonsingular transform. We show that the second-order Düffing system can be synchronized with the canonical projection of a third-order resemblance Lorenz system, and a third-order chaotic oscillator can be synchronized with the canonical projection of the resemblance cell neural network chaotic system . Numerical simulations have been carried out to illustrate our findings.

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