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The energy of Einstein–Maxwell dilation–axion black hole in the teleparallel geometry*

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The energy of the stationary axisymmetric Einstein–Maxwell dilation–axion (EMDA) black hole is studied in the context of the Hamiltonian formulation of the teleparallel equivalence of general relativity (TEGR). The energy expression for the finite and arbitrary space-like two spheres is exactly computed by means of the integral form of the constraint equations of the formalism naturally without any restriction on the metric parameters. We also show that our results give the same values obtained by other methods for some special cases.

Keywords: energy, the stationary axisymmetric EMDA black hole, teleparallel geometry
PACC: 0470, 9760, 0462

1. Introduction

Although the notion of energy-momentum localization has been considered since the advent of the general theory of relativity, the definition of energy-momentum density of the gravitational field is now still a controversial problem. By studying the properties of solutions of the Einstein field equations that describe the gravitational field of isolated material systems, it is concluded that a consistent expression for the energy density of the gravitational field could be given in terms of second-order derivatives of the metric tensor.[1−3] Möller[4] revived general relativity by constructing a new field theory and reconsidered a Lagrangian formulation for absolute parallelism. This gravitational theory is characterized by the vanishing curvature tensor (absolute parallelism) and the non-vanishing torsion tensor.[5−7] Then, some authors established the teleparallel theory of gravity,[8−20] i.e. teleparallel equivalence of general relativity (TEGR), and attempted to obtain the Hamiltonian formulation and energy-momentum density of the gravitational field.

The TEGR is formulated on the Weitzenböck spacetime,[21] which is just an alternative geometrical formulation of general relativity (GR) and therefore it provides an alternative insight into the theory. In TEGR, the dynamical field quantities correspond to orthogonal tetrad field $e_{a\mu}$ (a and μ are SO(3, 1) and spacetime indices, respectively) and the affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu} = e^{a\lambda}\partial_{\mu}e_{a\nu}$. Then, one can construct the Lagrangian density of the TEGR and establish a simple definition[11−14,22−24] for the energy of the gravitational field in the framework of Schwinger's time gauge condition.[25] Recently, by using this definition Rocha-Neto and Castello-Branco calculated the gravitational energy of Kerr and Kerr anti-de Sitter spacetimes.[26]

In this paper we apply the definition of the gravitational energy to calculate the energy of the stationary axisymmetric Einstein–Maxwell dilation–axion (EMDA) black hole enclosed by an arbitrary two-sphere. Furthermore, to compare with known results obtained by using other method, we discuss the gravitational energy in some special cases, such as $a \to 0$, $r \to \infty$, $r = r_{+}$ and slow rotation approximation.

Notation: spacetime indices $\mu, \nu, \ldots$ and SO (3, 1) indices $a, b, \cdots$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i; a = (0), (i)$. The tetrad field $e^{a}{}_{\mu}$ yields the definition of the torsion tensor: $T^{a}{}_{\mu\nu} = \partial_{\mu}e^{a}{}_{\nu} - \partial_{\nu}e^{a}{}_{\mu}$. The flat Minkowski

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spacetime metric is fixed by $\eta_{ab} = e_{a\mu}e_{b\nu}g^{\mu\nu} = (-,+,+,+)$. The tangent space indices are raised and lowered with the Lorentzian metric η_{ab}, while the spacetime indices are raised and lowered with the Riemannian metric $g^{\mu\nu}$.

2. The Weitzenböck spacetime

According to TEGR, the fundamental field variable describing gravity is a quadruple of parallel tetrad field $e_{a\mu}$ which is on the Weitzenböck spacetime. This spacetime is derived from the Riemann–Cartan type spacetime.

The Riemann–Cartan spacetime[5,7] is endowed with a metric $g_{\mu\nu}$ and a connection $\Gamma^{\lambda}_{\mu\nu}$, which is characterized by

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^{\rho}_{\mu\lambda}g_{\rho\nu} - \Gamma^{\rho}_{\nu\lambda}g_{\mu\rho} = 0, \quad (1)$$

$$\Gamma^{\lambda}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + K^{\lambda}{}_{\mu\nu}, \quad (2)$$

where the Christoffel connection $\left\{{\lambda \atop \mu\nu}\right\}$ and contortion tensor $K^{\lambda}{}_{\mu\nu}$ are

$$\left\{{\lambda \atop \mu\nu}\right\} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}),$$

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(T^{\lambda}{}_{\mu\nu} + T_{\mu}{}^{\lambda}{}_{\nu} - T_{\nu}{}^{\lambda}{}_{\mu}). \quad (3)$$

The torsion tensor and the curvature tensor are defined as

$$T^{\lambda}{}_{\mu\nu}(\Gamma) = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = -T^{\lambda}{}_{\nu\mu},$$

$$R^{\alpha}{}_{\beta\mu\nu}(\Gamma) = \partial_\mu\Gamma^{\alpha}_{\beta\nu} - \partial_\nu\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\lambda\mu}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\mu}. \quad (4)$$

The Riemann–Cartan spacetime has both nonzero curvature and torsion tensor. There exists two geometrical structure for the spacetime. Setting the torsion tensor vanish ($T^{\lambda}{}_{\mu\nu}(\Gamma)$=0), we will obtain the Riemann spacetime where the affine connection reduces to the Christoffel connection, or, we will obtain the Weitzenböck spacetime by letting the curvature vanish ($R^{\alpha}{}_{\beta\mu\nu}(\Gamma)$=0).

On the Weitzenböck spacetime the covariant derivative $\tilde{D}_\nu$ satisfies

$$\tilde{D}_\mu e_a{}^{\lambda} = \partial_\mu e_a{}^{\lambda} + \tilde{\Gamma}^{\lambda}_{\mu\nu}e_a{}^{\nu} = 0, \quad (5)$$

and the affine connection reduces to

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = -e^a{}_\nu\partial_\mu e_a{}^{\lambda} = e_a{}^{\lambda}\partial_\mu e^a{}_\nu. \quad (6)$$

Therefore, in this spacetime, the torsion tensor is

$$T^{\lambda}{}_{\mu\nu}(\tilde{\Gamma}) = \tilde{\Gamma}^{\lambda}_{\mu\nu} - \tilde{\Gamma}^{\lambda}_{\nu\mu} = e^{a\lambda}(\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}). \quad (7)$$

3. The gravitational energy expression of the TEGR

The Riemann–Cartan geometry is the basis of Poincaré gauge theory (PGT).[10,20] In this gauge field, the usual definitions of the curvature and torsion tensor are yielded by using the tetrad field $e_{a\mu}$ and the spin affine connection $\omega_{\rho ab}$,

$$R^a{}_{b\mu\nu} = \partial_\mu\omega_\nu{}^a{}_b - \partial_\nu\omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c\omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c\omega_\mu{}^c{}_b, \quad (8)$$

$$T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega_\mu{}^a{}_b e^b{}_\nu - \omega_\nu{}^a{}_b e^b{}_\mu. \quad (9)$$

The spacetime connection $\Gamma^{\lambda}_{\mu\nu}$ and the spacetime metric $g_{\mu\nu}$ are defined by

$$e^a{}_\lambda\Gamma^{\lambda}_{\mu\nu} = \partial_\mu e^a{}_\nu + \omega_\mu{}^a{}_b e^b{}_\nu, \quad (10)$$

$$e^a{}_\lambda\left\{{\lambda \atop \mu\nu}\right\} = \partial_\mu e^a{}_\nu + {}^o\omega_\mu{}^a{}_b e^b{}_\nu, \quad (11)$$

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu\eta_{ab}. \quad (12)$$

For an arbitrary spin connection, there exists an identity

$$\omega_{\rho ab} = {}^o\omega_{\rho ab} + K_{\rho ab} = -\omega_{\rho ba}, \quad (13)$$

where ${}^o\omega_{\rho ab}$ is the Levi-Civita connection which is metric compatible and torsion free; and the contortion tensor $K_{\rho ab}$ is

$$K_{\rho ab} = \frac{1}{2}e_a{}^\mu e_b{}^\nu K_{\rho\mu\nu} = \frac{1}{2}e_a{}^\mu e_b{}^\nu(T_{\rho\mu\nu} + T_{\mu\rho\nu} - T_{\nu\rho\mu}).$$

It is convenient to define a useful tensor Λ^{abc} as

$$\Lambda^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c), \quad (14)$$

where $T_b = T^a{}_{ab}$, $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$, $T_{abc} = e_b{}^\mu e_c{}^\nu T_{a\mu\nu}$. Then, we have

$$\Lambda^{abc}T_{abc} = \frac{1}{4}T^{abc}T_{abc} + \frac{1}{2}T^{abc}T_{bac} - T^aT_a. \quad (15)$$

Denoting the scalar curvature $R(e,\omega) = e^{a\mu}e^{b\nu}R_{ab\mu\nu}$ and substituting the arbitrary spin connection $\omega_{\rho ab}$ into the scalar curvature tensor, we obtain an identity[10,19]

$$eR(e,\omega) = eR(e) + e\Lambda^{abc}T_{abc} - 2\partial_\mu(eT^\mu), \quad (16)$$

where $R(e)$ is constructed out of $e^a{}_\mu$ only. This identity is one of bases of the equivalence of TEGR with Einstein's general relativity. The Einstein-Hilbert Lagrangian is

$$L_{\rm GR} = keR(e) = keR(e,\omega) - ke\Lambda^{abc}T_{abc},$$

where $2\partial_\mu(eT^\mu)$ is a total derivative term and is neglected, and $k = c^4/16\pi G$ (G is the gravitational constant).

In the TEGR, we require $R_{ab\mu\nu}(\omega) = 0$. Then we obtain

$$\begin{aligned}\Gamma^\lambda_{\mu\nu} \to \tilde{\Gamma}^\lambda_{\mu\nu} =& e_a{}^\lambda \partial_\mu e^a{}_\nu = \left\{{\lambda \atop \mu\nu}\right\} - {}^o\omega_{\mu ab} e^{a\lambda} e^b{}_\nu,\\ {}^o\omega_{\rho ab} =& -K_{\rho ab},\\ T^a{}_{\mu\nu} =& \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu,\\ L_{\rm GR} =& -keA^{abc}T_{abc}.\end{aligned}$$

Considering a class of Lagrangian quadratic in the torsion[19,20]

$$L = e(L_T + aR^{ab\mu\nu}R_{ab\mu\nu} + L_M), \tag{17}$$

where $L_T = kR(e,\omega) - kA^{abc}T_{abc}$, and L_M is the Lagrangian of matter fields, thus the Lagrangian L_T is an alternate "more geometric" interpretation of L_M.

Define the canonical stress-energy tensor and spin density as

$$T^a{}_\mu = \frac{1}{2e}\frac{\delta \mathcal{L}_M}{\delta e_a{}^\mu}, \qquad S^\rho{}_{ab} = \frac{1}{e}\frac{\delta \mathcal{L}_M}{\delta \omega_\rho{}^{ab}},$$

where $\mathcal{L}_M = eL_M$, the field equation can be written as[20]

$$G_{\mu\nu}(\{\}) = \tau_{\mu\nu} + T_{\mu\nu}, \tag{18}$$

$$D_\lambda R^{ab\rho\lambda} = S^{\rho ab}, \tag{19}$$

where $\tau_{\mu\nu} = -2(aR^{ab}{}_{\rho\mu}R_{ab}{}^\rho{}_\nu - \frac{1}{4}aR^{ab}{}_{\rho\lambda}R_{ab}{}^{\rho\lambda})$, and D_λ is the covariant derivative that acts with $\omega^\rho{}_{ab}$ on the tangent space indices and with $\left\{{\lambda \atop \mu\nu}\right\}$ on the spacetime indices. We can refer to other forms of the field equation.[5,7,10,18]

In the absence of spinning sources, the classical scalar matter couples to the metric alone. Thus in the framework of the teleparallel geometry in PGT, we have $R_{ab}{}^{\rho\lambda} = 0$(namely, fix the Poincaré gauge in terms of $\omega_\rho{}^{ab} = 0$) as the ground state solution and can obtain $\tau_{\mu\nu} = 0$; and Eq.(18) is reduced to the Einstein equation of GR, L_T to $L_T = -kA^{abc}T_{abc}$, and L to $L(e_{a\mu}) = -keA^{abc}T_{abc} + L_M$. Namely, the TEGR is equivalent to the general relativity.

References [11] and [12] show that the momentum canonically conjugated to $e_{a\mu}$ is

$$\Pi^{ak} = -4keA^{a0k}, \tag{20}$$

and the full expression of Π^{ak} can be written as

$$\begin{aligned}\Pi^{ak} =& ke[g^{00}(-g^{kj}T^a{}_{0j} - e^{aj}T^k{}_{0j}\\ &+ 2e^{ak}T^j{}_{0j}) + g^{0k}(g^{0j}T^a{}_{0j} + e^{aj}T^0{}_{0j})\\ &+ e^{a0}(g^{0j}T^k{}_{0j} + g^{ki}T^0{}_{0j})\\ &- 2(e^{a0}g^{0k}T^j{}_{0j} + e^{ak}g^{0j}T^0{}_{0j}) - g^{0i}g^{kj}T^a{}_{ij}\\ &+ e^{ai}(g^{0j}T^k{}_{ij} - g^{kj}T^0{}_{ij})\\ &- 2(g^{0i}e^{ak} - g^{ik}e^{a0})T^j{}_{ji}].\end{aligned} \tag{21}$$

Therefore, in the time gauge condition, the expression of field energy can be written as[11,12]

$$E = P^{(0)} = -\int_V \mathrm{d}^3x \partial_i \Pi^{(0)i}. \tag{22}$$

In the macrophysical limit,[7] because spin is averaged out in the large, the dynamical characterization of a continuous distribution of macroscopic matter can be successfully achieved by the energy-momentum ("momentum current") alone. The energy-momentum tensor of matter is the source of the metric tensor $g_{\mu\nu}$ of a Riemannian spacetime continuum. The momentum (Π^{ak}) canonically conjugated to $e_{a\mu}$ is described only in terms of the spacetime structure. In the absence of spinning sources, since TEGR and GR is equivalent, we can pick out a set of tetrad $e_{a\mu}$ corresponding to spacetime metric $g_{\mu\nu}$. Then, we can understand Eq.(22) as the total, gravitational and matter fields energy.[13,14] We will apply it to compute the energy of black hole over finite space volumes, which can be transformed into a surface integral.

4. The energy of EMDA black hole

The stationary solution for axisymmetric EMDA black hole is given by[27,28]

$$\begin{aligned}\mathrm{d}s^2 =& -\frac{\Delta - a^2\sin^2\theta}{\rho^2}\mathrm{d}t^2 - \frac{2Mar\sin^2\theta}{\rho^2}\mathrm{d}t\mathrm{d}\phi\\ &+ \frac{\rho^2}{\Delta}dr^2 + \rho^2\mathrm{d}\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}\mathrm{d}\phi^2,\end{aligned} \tag{23}$$

with

$$\begin{aligned}\Delta &= r^2 - 2mr + a^2,\\ \rho &= \sqrt{r^2 - 2Dr + a^2\cos^2\theta},\\ \Sigma &= \sqrt{\Gamma^2 - \Delta a^2\sin^2\theta},\\ \Gamma &= r^2 - 2Dr + a^2,\end{aligned} \tag{24}$$

where the charge is $Q = \sqrt{2\omega D(D-m)}$ (with $\omega = e^{2\Phi_0}$). The horizon is $r_\pm = (M - \frac{Q^2}{2\omega M} \pm$

$\sqrt{(M-\frac{Q^2}{2\omega M})^2-a^2)}=m\pm\sqrt{m^2-a^2}$ and the ADM mass is $M=m-D$. Here and hereafter we have taken $G=c=1$.

For a given spacetime metric tensor $g_{\mu\nu}$, there exists an infinite set of tetrad fields that yield $g_{\mu\nu}$. Since the EMDA spacetime is asymptotically flat, considering the Schwinger's time gauge condition,[12,26,29–31] a set of tetrads associated to the EMDA metric can be written as

$$e_{a\mu}=\begin{pmatrix} -A & 0 & 0 & 0 \\ B\sin\theta\sin\phi & C\sin\theta\cos\phi & Er\cos\theta\cos\phi & -Fr\sin\theta\sin\phi \\ -B\sin\theta\cos\phi & C\sin\theta\sin\phi & Er\cos\theta\sin\phi & Fr\sin\theta\cos\phi \\ 0 & C\cos\theta & -Er\sin\theta & 0 \end{pmatrix}, \tag{25}$$

where

$$A=\frac{\rho\sqrt{\Delta}}{\Sigma},\quad B=\frac{2Mar}{\rho\Sigma},\quad C=\frac{\rho}{\sqrt{\Delta}},$$
$$E=\frac{\rho}{r},\quad F=\frac{\Sigma}{\rho r}.$$

The components of the torsion tensor is presented in the Appendix. Thus, for an arbitrary space-like two-sphere, the field's energy $E_{\rm BH}$ of the EMDA black hole is given by

$$\begin{aligned} E_{\rm BH}=&P^{(0)}=-\int_S {\rm d}S_i \Pi^{(0)i} \\ =&-\int_S {\rm d}\theta{\rm d}\phi \Pi^{(0)1}(r,\theta,\phi) \\ =&\frac{1}{4}\left(\sqrt{\Gamma}+\frac{r^2-2Dr}{2a}\ln\frac{\sqrt{\Gamma}+a}{\sqrt{\Gamma}-a}\right) \\ &+\frac{1}{4}\left(\frac{2\sqrt{\alpha}}{{\rm i}a}-\frac{\sqrt{\alpha}\partial_r\Delta}{{\rm i}a\sqrt{\Delta}}\right) \\ &\times E\left(\frac{{\rm i}a}{\sqrt{r^2-2Dr}},\frac{\sqrt{r^2-2Dr}\sqrt{\Delta}}{\sqrt{\alpha}}\right) \\ &+\frac{1}{4}\left(\frac{\sqrt{\alpha}\partial_r\Delta}{{\rm i}a\sqrt{\Delta}}-\frac{\sqrt{\Delta}\partial_r\alpha}{{\rm i}a\sqrt{\alpha}}\right) \\ &\times F\left(\frac{{\rm i}a}{\sqrt{r^2-2Dr}},\frac{\sqrt{r^2-2Dr}\sqrt{\Delta}}{\sqrt{\alpha}}\right), \end{aligned} \tag{26}$$

where $\alpha=\Gamma^2-\Delta a^2, {\rm i}^2=-1$ and the elliptic functions $E(x,y)$ and $F(x,y)$ are defined as

$$E(x,y)=\int_0^x {\rm d}u\frac{\sqrt{1-y^2u^2}}{\sqrt{1-u^2}},$$
$$F(x,y)=\int_0^x {\rm d}u\frac{1}{\sqrt{1-u^2}\sqrt{1-y^2u^2}}.$$

Expression (26) is the major result of this paper.

Now, let us consider some special cases:

(1) In the asymptotic limit $r\to\infty$, from Eq.(26) we obtain

$$E_{\rm BH}=m-D=M.$$

This is just the Arnowitt–Deser–Misner (ADM) energy of the EMDA black hole.

(2) The energy of the static ($a=0$) dilation black hole can be easily found using Eq.(26),

$$E_{\rm BH}=\sqrt{r^2-2Dr}-(r-D)\sqrt{\frac{r-2m}{r-2D}}.$$

When $D=0$, the energy reduces to $E_{\rm BH}=r\left(1-\sqrt{1-\frac{2m}{r}}\right)$, which is the outer energy within an arbitrary space-like surface of fixed radius r in the Schwarzschild black hole.[29]

(3) When $r=r_+=m+\sqrt{m^2-a^2}$, noting

$$\Delta=0,\qquad \Gamma=2Mr_+,$$
$$\alpha=4M^2r_+^2,\quad r_+^2-2Dr_+=2Mr_+-a^2,$$

we find

$$\begin{aligned} E_{\rm BH}=&\frac{\sqrt{2Mr_+}}{4}+\frac{M\sqrt{2mr_+-a^2}}{a} \\ &\times Arcsinh\left[\frac{a}{r_+^2-2Dr_+}\right] \\ &+\frac{2Mr_+-a^2}{8a}\ln\frac{a+\sqrt{2r_+M}}{a-\sqrt{2r_+M}}, \end{aligned} \tag{27}$$

and $E_{\rm BH}(r_+,a\to 0)=2\sqrt{mM}$ which is the irreducible mass M_{irr} of the static EMDA black hole.

(4) If we consider the slow rotation approximation (namely, $\frac{a}{r}<<1$), we have

$$\begin{aligned} E_1=&\frac{1}{4}\int\left(\rho+\frac{\Sigma}{\rho}\right)\sin\theta{\rm d}\theta \\ =&\sqrt{r^2-2Dr}+\frac{a^2}{3\sqrt{r^2-2Dr}} \\ &\times\left(1+\frac{m-D}{r-2D}\right)+O\left(\frac{a^4}{r^4}\right), \end{aligned}$$

$$E_2 = \frac{1}{4}\int -\frac{\sqrt{\Delta}\partial_r \Sigma}{\rho}\sin\theta \mathrm{d}\theta$$
$$= -(r-D)\frac{\sqrt{\Delta}}{\sqrt{r^2-2Dr}} + \frac{a^2}{6r\sqrt{r^2-2Dr}}$$
$$\times\left[\frac{r-D}{r-2D} + \frac{2r(m-D)}{(r-2D)^2}\right] + O\left(\frac{a^4}{r^4}\right).$$

Then, the energy is given by

$$E_{\rm BH} = E_1 + E_2$$
$$\approx \left[\sqrt{r^2-2Dr} - (r-D)\frac{\sqrt{\Delta}}{\sqrt{r^2-2Dr}}\right]$$
$$+ \frac{a^2}{6\sqrt{r^2-2Dr}}\left\{2 + \frac{2(m-D)}{r-2D}\right.$$
$$\left. + \frac{\sqrt{\Delta}}{r}\left[\frac{r-D}{r-2D} + \frac{2r(m-D)}{(r-2D)^2}\right]\right\}, \quad (28)$$

where the terms of order $O\left(\frac{a^4}{r^4}\right)$ and higher are neglected. This result was also obtained in Ref.[32] by calculating Brown-York quasilocal energy. It is interesting to note that the term E_1 corresponds exactly to $-\varepsilon_0$ in Ref.[32], and the term E_2 to ε in Ref.[32].

For $D = 0$, Eq.(28) gives the same expression as Eq.(15) of the Kerr black hole in Ref.[30].

5. Conclusion

We have studied the energy of the stationary axisymmetric EMDA black hole in terms of the Hamiltonian formulation of the TEGR. The energy expression (26) for finite and arbitrary space-like two-sphere is exactly worked out by means of the integral form of the constraint equations of the formalism naturally without any restriction on the metric parameters. In order to compare the results obtained from Eq.(26) with the known results reported in the literature, we also reduce the energy (26) into some special cases, such as $a \to 0$, $r \to \infty$, $r = r_+$ and slow rotation approximation, which show that these results exactly agree with those obtained by using other methods.

Appendix: Calculation of torsion and momenta

Now we present the nonvanishing components of the torsion tensor related to the set of tetrads given in Eq.(25) as follows:

$$T_{(0)01} = \partial_r A,$$
$$T_{(0)02} = \partial_\theta A,$$
$$T_{(1)01} = -2Ma\sin\theta\sin\phi\ \partial_r\left(\frac{r}{\rho\Sigma}\right),$$
$$T_{(1)02} = -2Mar\sin\phi\ \partial_\theta\left(\frac{\sin\theta}{\rho\Sigma}\right),$$
$$T_{(1)03} = -\frac{2Mar}{\rho\Sigma}\sin\theta\cos\phi,$$
$$T_{(1)12} = \cos\theta\cos\phi\ \partial_r\ \rho\ - cos\phi\ \partial_\theta\ (\rho\sin\theta/\sqrt{\Delta}),$$
$$T_{(1)13} = \sin\theta\sin\phi\left[\frac{\rho}{\sqrt{\Delta}} - \partial_r\left(\frac{\Sigma}{\rho}\right)\right],$$
$$T_{(1)23} = \rho\cos\theta\sin\phi - \sin\phi\ \partial_\theta\left(\frac{\Sigma\sin\theta}{\rho}\right),$$
$$T_{(2)01} = 2Ma\sin\theta\cos\phi\ \partial_r\left(\frac{r}{\rho\Sigma}\right),$$
$$T_{(2)02} = 2Mar\cos\phi\ \partial_\theta\left(\frac{\sin\theta}{\rho\Sigma}\right),$$
$$T_{(2)03} = \frac{-2Mar}{\rho\Sigma}\sin\theta\ \sin\phi,$$
$$T_{(2)12} = \cos\theta\sin\phi\ \partial_r\ \rho - \sin\phi\ \partial_\theta\ (\rho\sin\theta/\sqrt{\Delta}),$$
$$T_{(2)13} = -\sin\theta\cos\phi\left[\frac{\rho}{\sqrt{\Delta}} - \partial_r\left(\frac{\Sigma}{\rho}\right)\right],$$
$$T_{(2)23} = \cos\phi\ \partial_\theta\left(\frac{\Sigma\sin\theta}{\rho}\right) - \rho\cos\theta\cos\phi,$$
$$T_{(3)12} = -\sin\theta\ \partial_r\ \rho - \partial_\theta\ (\rho\cos\theta/\sqrt{\Delta}).$$

Then, using Eq.(21) we find that the component $\Pi^{(0)1}$ of the canonical momentum for the EMDA black hole can be expressed as

$$\Pi^{(0)1} = -2k\sin\theta\left(\rho + \frac{\Sigma}{\rho} - \frac{\sqrt{\Delta}}{\rho}\partial_r\Sigma\right).$$

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