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# Interaction between two folded solitary waves for a modified Broer–Kaup system*

Wu Xiao-Fei(吴晓飞)[a)†], Zhu Jia-Min(朱加民)[a)], and Ma Zheng-Yi(马正义)[a)b)]

[a)] *Department of Physics, Zhejiang Lishui University, Lishui 323000, China*

[b)] *Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, China*

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By means of a Painlevé–Bäcklund transformation and a multi-linear separation-of-variable approach, abundant localized coherent excitations of a modified Broer–Kaup system are derived. There appear possible phase shifts for the interactions of the (2+1)-dimensional novel localized structures, which are discussed in this paper.

**Keywords:** modified Broer–Kaup system, Bäcklund transformation, separation-of-variable method, folded localized excitations

**PACC:** 0340, 0365, 0545

## 1. Introduction

Modern soliton theory is widely applied in almost all of the physics fields, such as fluid mechanics, plasma physics, nonlinear optics, condensed matter physics, etc.[1] Particularly, some localized coherent theories of (2+1)-dimensional soliton systems have proved to be of considerable interest. For instance, a discovery, via the Bäcklund transformation by Boiti *et al*,[2] of dromion type of coherent solutions of the Daver–Stewartson I system aroused renewed interest in (2+1)-dimensional soliton systems. In the recent decade, the main effort to study higher-dimensional physical models has focused on single valued localized excitations, such as solitoffs, dromions, rings, lumps, breathers, instantons, peakons, compactons, localized chaotic and fractal patterns, etc.[3−8] However, in reality there are various complicated phenomena like the ordinary bubbles on a fluid surface, which are folded in a simple way but they cannot be described by the single valued functions obviously, neither can the various types of ocean waves. In Refs. [9,10], some multi-valued localized excitations were introduced to describe the folded solitary wave and define a new type of soliton, i.e. foldon, if the interaction between the folded solitary waves (FSWs) is fully elastic. Actually, the simple foldons in lower dimensions can be equivalently called the loop solitons which can be found in many (1+1)-dimensional integrable models[11,12] and have been applied in some physical branches like the quantum theory, string theory and particle physics.[13−16] However, beyond these lower-dimensional foldons, we know little about some foldons in higher dimensions.

Inspired by the foldon phenomena, we focus on the interaction between two FSWs for the (2+1)-dimensional modified Broer–Kaup system (mBKs) where exists the elastic or nonelastic behaviour.

For the (2+1)-dimensional modified Broer–Kaup system (mBKs),[17] we have

$$(H_t - H_{xx} + 2HH_x)_y + G_{xx} + \alpha G_x = 0, \quad (1)$$

$$G_t + 2(GH)_x + G_{xx} + 4\alpha(G_x - H_{xy}) + \beta(G - 2H_y) = 0, \quad (2)$$

where $\alpha$, $\beta$ are arbitrary constants. Zhang *et al*[18] first extended the (1+1)-dimensional Broer–Kaup (BK) system[19,20]

$$u_t = uu_x + v_x - \frac{1}{2}u_{xx}, \quad (3)$$

$$v_t = (uv)_x + \frac{1}{2}v_{xx}, \quad (4)$$

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†E-mail: xfwu66@163.com

to a new (2+1)-dimensional nonlinear evolution equation (which is called (2+1)-dimensional generalized Broer–Kaup system)

$$h_t - h_{xx} + 2hh_x + u_x + Au + Bg = 0, \tag{5}$$

$$g_t + 2(gh)_x + g_{xx} + 4A(g_x - h_{xy}) + 4B(g_y - h_{yy}) + C(g - 2h_y) = 0, \tag{6}$$

$$u_y - g_x = 0, \tag{7}$$

by means of the Painlevé analysis, and gave some exact particular solutions of Eqs.(5)–(7) by using the truncated Painlevé expansion; Huang *et al*[17] then derived the mBKs through differentiating Eq.(5) with respect to the variable $y$ once and substituting Eq.(7) into Eq.(5) ($A = \alpha, B = 0, C = \beta, h = H, g = G$), and applied the proposed variable-coefficient projective Riccati equation method to finding many exact soliton-like solutions for it. However, to the best of our knowledge, the abundant localized coherent structures for the (2+1)-dimensional mBKs, especially its multi-valued (folded) localized excitations were not reported in the preceding literature.

The present paper falls into four parts. In Sec.2, after using the extended homogeneous balance method, both Bäcklund transformation and rather a general separation-of-variable solution of the (2+1)-dimensional mBK system are found. In Sec.3, several interactions of elastic and nonelastic behaviours between two FSWs are discussed. Finally, the section 4 contains a short summary of the results.

## 2. Separation-of-variable excitations for the (2+1)-dimensional mBKs

Now we apply the separation-of-variable approach to Eqs.(1) and (2).

Firstly, we apply the following Painlevé–Bäcklund transformation to $H, G$ in Eqs.(1) and (2)

$$H = \sum_{j=0}^{\alpha_1} H_j f^{j-\alpha_1}, \quad G = \sum_{j=0}^{\alpha_2} G_j f^{j-\alpha_2}, \tag{8}$$

where $H_{\alpha_1}$ and $G_{\alpha_2}$ are the arbitrary seed solutions of the (2+1)-dimensional mBks. By using the leading term analysis, we obtain

$$\alpha_1 = 1, \quad \alpha_2 = 2. \tag{9}$$

Substituting Eqs.(8) and (9) directly into Eqs.(1) and (2) and considering the fact that the functions $H_1$ and $G_2$ are the seed solutions of the original model, yield

$$\sum_{i=0}^{3} P_{1i} f^{i-4} = 0, \quad \sum_{i=0}^{3} P_{2i} f^{i-4} = 0, \tag{10}$$

where $P_{1i}$ and $P_{2i}$ are the functions of $\{H_j, G_j, f\}$ and their derivatives. Due to the complexity of the expressions of $P_{1i}$ and $P_{2i}$, we neglect their concrete forms. Eliminating the leading and sub-leading terms in Eq.(10), the functions $\{H_j, G_j, j \le 2\}$ are determined. Inserting all results into Eq.(8) and rewriting its form, the Painlevé–Bäcklund transformation becomes the following form:

$$H = \ln(f)_x + H_1, \quad G = \ln(f)_{xy} + G_2. \tag{11}$$

For the convenience of discussion, we choose the seed solutions $H_1$ and $G_2$ as

$$H_1 = H_0(x,t), \quad G_2 = 0, \tag{12}$$

where $H_0(x,t)$ is an arbitrary function of indicated arguments.

Now substituting Eq.(11) with the seed solutions (12) into Eqs.(1) and (2) yields an identical trilinear equation of $f$ under the above transformation,

$$\begin{aligned}&(f_{xxxy} + 2\alpha f_{xxy} + f_{xyt} + 2H_0 f_{xxy} + 2H_{0x} f_{xy}) f^2 \\ &- (2\alpha f_{xx} f_y + 4\alpha f_x f_{xy} + f_{xy} f_t \\ &+ f_x f_{xxy} + f_{xt} f_y + f_x f_{yt} + f_{xxx} f_y + f_{xx} f_{xy} \\ &+ 2 f_x f_y H_{0x} + 2H_0 f_{xx} f_y + 4H_0 f_x f_{xy}) f \\ &+ 2 f_{xx} f_x f_y + 4\alpha f_x^2 f_y + 4H_0 f_x^2 f_y + 2 f_x f_y f_t = 0.\end{aligned} \tag{13}$$

Secondly, after obtaining the trilinear equation of the original system, we select an appropriate hypothesis for separating the variables in the function $f$. From among available models, we choose a modifying Horita's multi-soliton form. For instance, we often take $f$ as an ansatz for some significant physical models[21–23]

$$\begin{aligned}f(x,y,t) =& a_0 + a_1 p(x,t) + a_2 q(y,t) \\ &+ a_3 p(x,t) q(y,t),\end{aligned} \tag{14}$$

where the separation-of-variable functions $p \equiv p(x,t)$ and $q \equiv q(y,t)$ are only the functions of $(x,t)$ and $(y,t)$ respectively; $a_0$, $a_1$, $a_2$ and $a_3$ are arbitrary constants. When $p$ and $q$ are set as exponential functions, Eq.(14) is just the Hirota's two-soliton form.

Inserting the ansatz (14) into Eq.(13) and carrying out some tedious calculations carefully, we obtain the following simple equations with variables separated:

$$p_t + 2\alpha p_x + p_{xx} + 2H_0 p_x + c_0(a_2 + a_3 p) = 0, \quad (15)$$

$$q_t = c_0(a_1 + a_3 q), \quad (16)$$

where $c_0 \equiv c_0(t)$ is an arbitrary function of time $t$.

Obviously, Eq.(16) has the solution

$$q = \frac{Q(y)}{a_3}(e^{\int a_3 c_0 dt} - a_1), \quad (17)$$

where $e^{\int a_3 c_0 \mathrm{d}t} - a_1 \neq 0$ and $Q(y)$ is an arbitrary function of $y$. Although it is not easy to obtain a general solution of Eq.(15) for any fixed $H_0$, we can treat the problem in an alternative way. Since $H_0$ is an arbitrary seed solution, we can view $p$ as an arbitrary function of $x$ and $t$, the seed solution $H_0$ can be fixed by Eq.(15), which reads

$$H_0 = -\frac{1}{2p_x}[p_t + 2\alpha p_x + p_{xx} + c_0(a_2 + a_3 p)]. \quad (18)$$

Finally, substituting Eq.(14) with the seed solution (18) into Eq.(11), we obtain quite a general excitation of the (2+1)-dimensional mBKs

$$H = \frac{(a_1 + a_3 q)p_x}{a_0 + a_1 p + a_2 q + a_3 pq} - \frac{1}{2p_x}[p_t + 2\alpha p_x + p_{xx} + c_0(a_2 + a_3 p)], \quad (19)$$

$$G = \frac{2(a_0 a_3 - a_1 a_2)p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \quad (20)$$

where $p$ is an arbitrary function of $x$ and $t$, $q$ is determined by Eq.(17). It is interesting to mention that performed slightly by scalar transformation, the expression (20) can be valid for many (2+1)-dimensional soliton models, such as the Davey–Stewartson (DS) equation, the asymmetric DS (ADS) equation, the Nizhnik–Novikov–Veselov (NNV) equation, the asymmetric NNV (ANNV) equation, the Burgers equation, the Breaking soliton system, the Ablowitz–Kaup–Newell–Segur (AKNS) system, the generalized AKNS (GAKNS) system, the generalized nonlinear Schrödinger (GNLS) equation, the dispersive long wave (DLW) equation, and so on.[24–27] Therefore, we refer to the expression (20) as a common field quantity.

In fact, we can expand the ansatz (14) into

$$F(x, y, t) = a_0 + \sum_{i=1}^{L} a_i p_i(x, t) q_i(y, t), \quad (21)$$

where $L$ is an arbitrary integer constant, $a_0, a_1, \cdots, a_L$ are arbitrary constants and the separation-of-variable functions $p_i \equiv p_i(x, t)$ and $q_i \equiv q_i(y, t)(i = 1, 2, \cdots, L)$ are only the functions of $(x, t)$ and $(y, t)$ respectively. When we take $L = 3$, $q_1 = p_2 = 1$, the above function $F(x, y, t)$ is just the same as the function $f(x, y, t)$ in Eq.(14).

Substituting the ansatz (21) into Eq.(13), we obtain

$$H_0 = -\frac{c(t)p_i + p_{it} + 2\alpha p_{ix} + p_{ixx}}{2p_{ix}},$$
$$q_i = Q_i(y)\mathrm{e}^{\int c(t)\mathrm{d}t}, (i = 1, 2, \cdots, L), \quad (22)$$

where $c(t)$ is an arbitrary function of $t$ and $Q_i(y)(i = 1, 2, \cdots, L)$ are arbitrary functions of $y$. In this way, we obtain the separation-of-variable solution of the (2+1)-dimensional mBKs as follows:

$$H = \frac{\sum_{i=1}^{L} a_i p_{ix} q_i}{F(x, y, t)} + H_0,$$

$$G = \frac{1}{F(x, y, t)^2}\left(\sum_{i=1}^{L} a_i(a_0 q_{iy} + \sum_{j=1, j\neq i}^{L} a_j p_j(q_{iy} q_j - q_i q_{jy}))p_{ix}\right). \quad (23)$$

In the above solutions, the functions $F(x, y, t), H_0$ and $q_i(i = 1, 2, \cdots, L)$ are expressed by the expressions (21) and (22) respectively; $p_i(i = 1, 2, \cdots, L)$ are arbitrary functions of $(x, t)$.

## 3. Some novel localized structures for the (2+1)-dimensional mBKs

In this section, we discuss some novel localized structures for the field $G$ expressed by the expression (20). As the arbitrariness of either the function $p$ or the function $q$ is included in the expression (20), the field $G$ may obviously extend to several different types of structures, such as the multi-soliton solutions, multi-dromion and dromion lattices solutions, multiple ring soliton solutions, breathers, instantons, oscillating solitons, peakons, compactons, chaotic and fractal localized solutions. For instance, a dromion solution which is localized in all directions can be driven by multiple straight-line and straight-line ghost solitons. When the functions $p$ and $q$ are simply chosen as

$$p = \sum_{i=1}^{M} \exp(K_i x + L_i t + x_{0i}),$$

$$q = 1 + \sum_{i=1}^{N} \exp(k_i y + y_{0i}),$$

where $K_i$, $L_i$, $k_i$, $x_{0i}$ and $y_{0i}$ are arbitrary constants and $M$ and $N$ are positive integers, we can obtain a particular multi-dromion solution for the field $G$. Meanwhile, when functions $p$ and $q$ are considered to be some piecewise smooth functions, we can derive some multi-peakon excitations for the field $G$. For example,

$$p = \sum_{i=1}^{M} \begin{cases} \chi_i(x+kt), & x+kt \le 0, \\ -\chi_i[-(x+kt)] + 2\chi_i(0), & x+kt > 0, \end{cases}$$

$$q = \sum_{i=1}^{N} \begin{cases} \psi_i(y), & y \le 0, \\ -\psi_i(-y) + 2\psi_i(0), & y > 0, \end{cases}$$

where the functions $\chi_i(x+kt) \equiv \chi(\xi)$ and $\psi_i(y)$ are the differentiable functions of the indicated arguments and possess the boundary conditions

$$\chi_i(\pm\infty) = A_{\pm i}, \ (i = 1, 2, \cdots, M),$$

$$\psi_i(\pm\infty) = B_{\pm i}, \ (i = 1, 2, \cdots, N)$$

with $A_{\pm i}$ and $B_{\pm i}$ being constants and/or even infinity.

In order to construct kinds of interesting folded localized excitations and/or foldons for the field $G$, we should introduce some suitable multi-valued functions.[28] For example,

$$p_x = \sum_{j=1}^{M_1} P_j(\xi + c_j t),$$

$$x = \xi + \sum_{j=1}^{M_2} X_j(\xi + c_j t), \tag{24}$$

where $P_j$ and $X_j$ are localized excitations with properties $P_j(\pm\infty) = 0, X_j(\pm\infty) =$const. From Eq.(24), one knows that $\xi$ may be a multi-valued function in some suitable regions of $x$ by selecting the functions $X_j$ appropriately. Therefore, the function $p_x$, which is obviously an interaction solution of $M_1$ localized excitations since the property $\xi \mid_{x\to\infty} \to \infty$, may be a multi-valued function of $x$ in these areas though it is a single-valued function of $\xi$. Actually, most of the known multi-loop solutions are the particular situations of Eq.(24). Similarly, we also treat the function $q(y,t)$ in this way

$$q_y = \sum_{j=1}^{N_1} Q_j(\eta + \omega_j t),$$

$$y = \eta + \sum_{j=1}^{N_2} Y_j(\eta + \omega_j t). \tag{25}$$

Generally speaking, if the functions $p$ and $q$ are taken as multiple localized excitations that possess the phase shifts of (1+1)-dimensional models then the (2+1)-dimensional localized excitations involving the expression (20) inherit phase shift structures. For example, when setting

$$p_x = 0.8\mathrm{sech}^4\xi + 0.5\mathrm{sech}^2(\xi - 0.25t),$$
$$x = \xi - 1.5\tanh\xi - 1.5\tanh(\xi - 0.25t); \tag{26}$$

$$q_y = \mathrm{sech}^2\eta, \quad y = \eta - 2\tanh\eta + \tanh^3\eta,$$
$$(a_0 = 20, a_1 = a_2 = a_3 = 1), \tag{27}$$

we can obtain folded localized excitations that possess phase shifts. Some interesting structures are depicted in Fig.1–Fig.3. From Fig.1, one can find that the interaction between the two folded localized structures is fully elastic since the velocity of one of the localized structures has set to be zero and there are still phase shifts for the two localized structures. To see more carefully, one can easily find that the position located by the large static localized structure moves from about $x = -1.5$ to $x = 1.5$ and its shape is completely preserved after interaction.

Figure 2 is another interesting type of two folded localized excitations that inherit phase shift structures if we take

$$p_x = 0.8\mathrm{sech}^4\xi + 0.5\mathrm{sech}^2(\xi - 0.25t),$$
$$x = \xi - 1.5\tanh\xi - 1.5\tanh(\xi - 0.25t); \tag{28}$$

$$q_y = e^{2\eta}, \quad y = \eta - 2\tanh\eta, \tag{29}$$

and $a_0 = 20, a_1 = a_2 = a_3 = 1$.

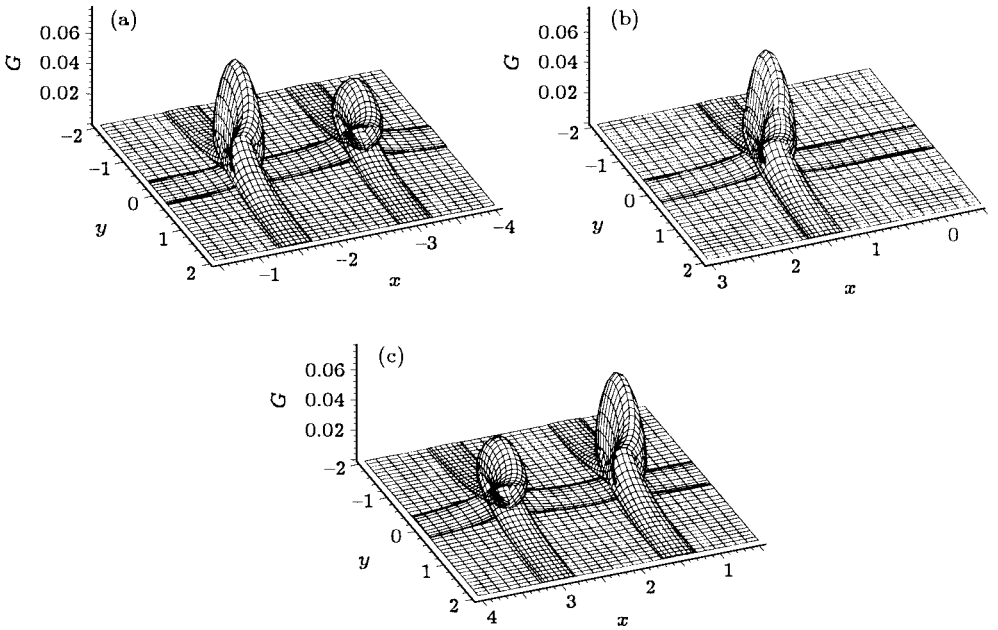



**Fig.1.** The evolution of the interaction between two folded localized structures for the field $G$ expressed by Eq.(20) with conditions (26) and (27) at different times (a) $t = -18$, (b) $t = 12$, (c) $t = 18$.

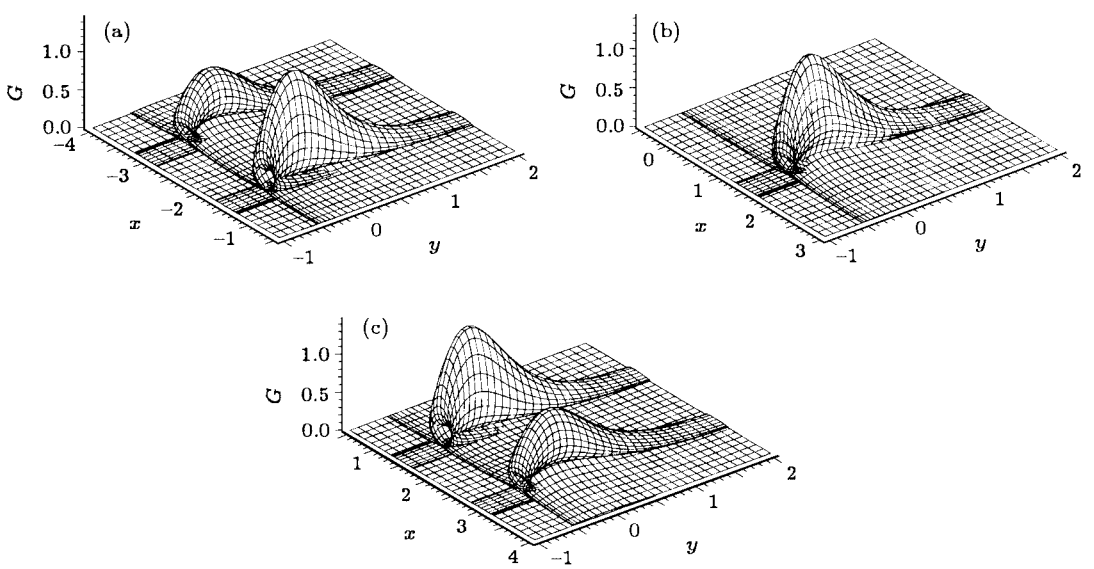



**Fig.2.** The evolution of the interaction between two folded localized structures for the field $G$ expressed by Eq.(20) with conditions (28) and (29) at different times (a) $t = -18$, (b) $t = 12$, (c) $t = 18$.

Along with the above ideas, however, we find that the interaction between folded localized structures may exhibit another novel property, which is not fully elastic for their shapes are not completely preserved after interaction. In order to clarify this phenomena more clearly and visually, a simple example is depicted in Fig.3 when

$$p_x = 10\mathrm{sech}^6(\xi - t) + 12\mathrm{sech}^2\xi,$$
$$x = \xi - 1.15\tanh\xi - 1.15\tanh(\xi - t); \quad (30)$$

$$q_y = 2\mathrm{sech}^2\eta, \quad y = \eta - 1.15\tanh\eta, \quad (31)$$

and $a_0 = 30, a_1 = a_2 = a_3 = 1$.

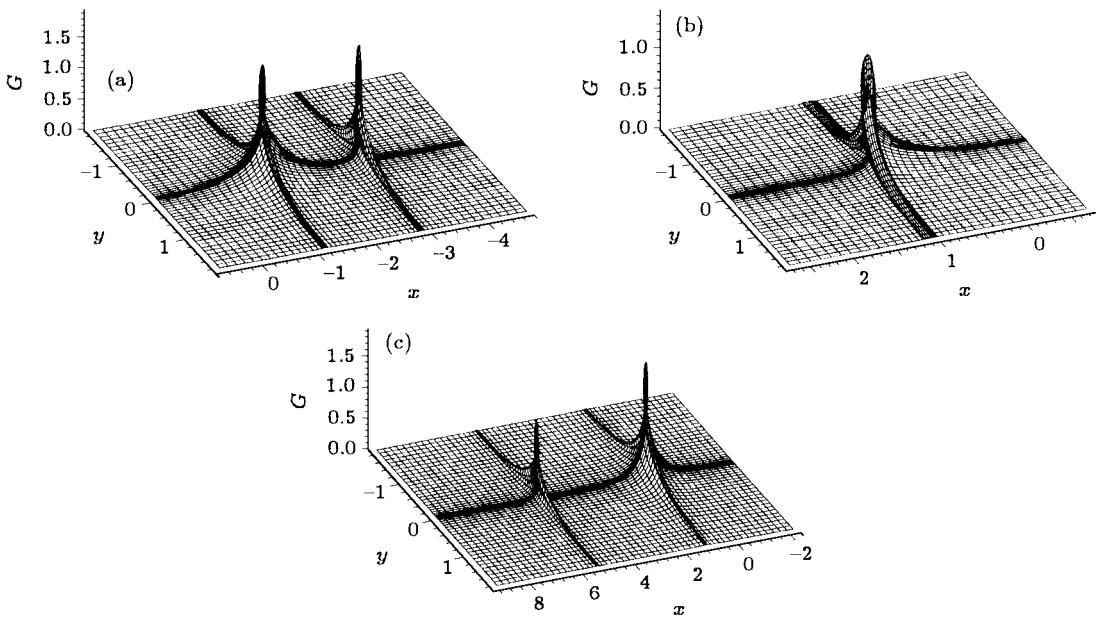



**Fig.3.** The evolution of the interaction between two folded localized structures for the field $G$ expressed by Eq.(20) with conditions (30) and (31) at different times (a) $t = -4$, (b) $t = 2.2$, (c) $t = 6.5$.

## 4. Summary and discussion

In summary, with the help of the Painlevé–Bäcklund transformation and using the multilinear separation-of-variable approach, the (2+1)-dimensional mBK system is solved. A general expression of excitation which includes two functions with variables separated for the (2+1)-dimensional mBK system is derived. Based on the general excitations with variables separated , various types of localized coherent soliton structures, such as multi-dromion, multi-ring, multi-lump solutions, breathers, instantons, peakons, chaotic and fractal soliton solutions, etc. can be constructed by selecting appropriate arbitrary functions. Furthermore, some possible phase shifts for the interactions of the (2+1)-dimensional localized structures are investigated. To our knowledge, the general localized excitations, especially the folded structures, for the (2+1)-dimensional mBK system have not reported yet in the previous literature. Due to the wide applications of the soliton theory in reality, it is worthy to learn more about the localized excitations and their applications.

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