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Methods of analytical mechanics for solving differential equations of first order*

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A differential equation of first order can be expressed by the equation of motion of a mechanical system. In this paper, three methods of analytical mechanics, i.e. the Hamilton–Noether method, the Lagrange–Noether method and the Poisson method, are given to solve a differential equation of first order, of which the way may be called the mechanical methodology in mathematics.

Keywords: analytical mechanics, differential equation, Noether theory, Poisson method
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1. Introduction

There are some classical methods of solving a differential equation of first order, such as the separation-of-variable method, the homogeneous equation, the linear equation, the Jacobi equation and the Riccati equation, etc.[1−5] In order to find the integrals of the differential equations of mechanical systems, a complete set of special integral theories and methods have been developed in analytical mechanics, such as the method of reduction of order, the canonical transformation, the Poisson method, the Hamilton–Jacobi method,[6,7] the field method,[8,9] etc.

Since Noether in 1918 revealed the inherent relation between symmetry and conserved quantity, the methods of finding the conserved quantities of mechanical systems by using the Noether theory have developed rapidly.[10−20] In this paper, three of the integral methods in analytical mechanics are introduced to solve a general differential equation of first order.

2. Hamilton–Noether method

Consider a differential equation of first order

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{g(x,y)}{f(x,y)}, \tag{1}$$

where f and g are continuous functions of x and y, and $f(x,y)\neq 0$.

For Eq.(1), introduce an intermediate variable t such that

$$\begin{aligned}\dot{x}&=f(x,y),\\ \dot{y}&=g(x,y),\end{aligned} \tag{2}$$

where

$$\dot{x}=\frac{\mathrm{d}x}{\mathrm{d}t},\quad \dot{y}=\frac{\mathrm{d}y}{\mathrm{d}t}. \tag{3}$$

It is easy to see that solutions of Eqs.(2) must be those of Eq.(1).

In the following, we use the Hamilton–Noether method to partly transform Eqs.(2) into Hamilton's equations of a mechanical system, and then find its conserved quantity by using the Noether theory.

Firstly, partly transform Eqs.(2) into Hamilton's equations of a mechanical system. Setting

$$q=x,\quad p=y, \tag{4}$$

and taking the Hamiltonian and generalized force as follows:

$$\begin{aligned}H&=p\int_0^1 f(q,\tau p)\mathrm{d}\tau,\\ Q&=\frac{\partial H}{\partial q}+g(q,p),\end{aligned} \tag{5}$$

respectively, Eqs.(2) can be expressed as

$$\dot{q}=\frac{\partial H}{\partial p},\quad \dot{p}=-\frac{\partial H}{\partial q}+Q. \tag{6}$$

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This is the Hamilton system with a nonconservative force.

Secondly, find the conserved quantity of the system (6) by using the Noether theory. The Noether identity yields[11,12]

$$p\dot{\xi}-\frac{\partial H}{\partial t}\xi_0-\frac{\partial H}{\partial q}\xi-H\dot{\xi}_0+Q(\xi-\dot{q}\xi_0)+\dot{G}_N=0, \quad (7)$$

where $\xi_0=\xi_0(t,q,p)$ and $\xi=\xi(t,q,p)$ are infinitesimal generators, $G_N=G_N(t,q,p)$ is the gauge function. The Noether theory shows that if the expression (7) has a solution, then the system (6) has the following Noether conserved quantity:

$$I_N=p\xi-H\xi_0+G_N=\text{const.} \quad (8)$$

Finally, we rewrite the expression (8) in coordinates x and y. Then the solution of Eqs.(2) can be obtained. It is also a solution of Eq.(1).

3. Lagrange–Noether method

Here we use the Lagrange–Noether method to partly transform Eqs.(2) into the Lagrange equation of a mechanical system, and then find its conserved quantity by using the Noether theory.

Firstly, express Eqs.(2) as a Lagrange system. Suppose that y can be solved from the first equation of Eqs.(2), and written as

$$y=h(x,\dot{x}). \quad (9)$$

Substituting the expression (9) into the second equation of Eqs.(2), we obtain

$$\frac{\partial h}{\partial x}\dot{x}+\frac{\partial h}{\partial \dot{x}}\ddot{x}=g(x,h(x,\dot{x})).$$

Suppose that $\ddot{x}$ can be solved, and written as

$$\ddot{x}=F(x,\dot{x}), \quad (10)$$

where

$$F(x,\dot{x})=\frac{g}{\dfrac{\partial h}{\partial \dot{x}}}-\frac{\dfrac{\partial h}{\partial x}}{\dfrac{\partial h}{\partial \dot{x}}}\dot{x}. \quad (11)$$

Equation (10) can be regarded as a holonomic mechanical system with one degree of freedom, and its Lagrangian and generalized force are

$$L=\frac{1}{2}\dot{x}^2-V(x), \quad Q=F(x,\dot{x})+\frac{\mathrm{d}V}{\mathrm{d}x}, \quad (12)$$

respectively, where $V(x)$ is an arbitrary smooth potential function.

Secondly, find the conserved quantity of the system (12) by using the Noether theory. The Noether identity gives[17–19]

$$L\dot{\xi}_0+\dot{x}(\dot{\xi}-\dot{x}\dot{\xi}_0)-\frac{\mathrm{d}V}{\mathrm{d}x}\xi+Q(\xi-\dot{x}\xi_0)+\dot{G}_N=0, \quad (13)$$

where $\xi_0=\xi_0(t,x,\dot{x})$ and $\xi=\xi(t,x,\dot{x})$ are infinitesimal generators, $G_N=G_N(t,x,\dot{x})$ is the gauge function. If the expression (13) has a solution, then, according to the Noether theorem, the Noether conserved quantity can be found as follows:

$$I_N=L\xi_0+\frac{\partial L}{\partial \dot{x}}(\xi-\dot{x}\xi_0)+G_N=\text{const.} \quad (14)$$

Finally, replacing the $\dot{x}$ appearing in the expression (14) by the first equation of Eqs.(2) yields the integral of Eqs.(2).

4. Poisson method

We employ the Poisson method to transform Eqs.(2) into the form of Lie or Lie-admissible algebra, and then find its integral by using the generalized Poisson condition.

Transforming Eqs.(2) into the form of Lie-admissible algebra and setting

$$a^1=q, \quad a^2=p,$$

Equations (6) can be expressed as[20,21]

$$\dot{a}^\mu-S^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0 \quad (\mu,\nu=1,2), \quad (15)$$

where

$$(S^{\mu\nu})=\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\begin{pmatrix}0&0\\0&\dfrac{Q}{\dfrac{\partial H}{\partial a^2}}\end{pmatrix}. \quad (16)$$

System (15) has an integral I if and only if

$$\frac{\partial I}{\partial t}+I\circ H=0, \quad (17)$$

where

$$I\circ H=\frac{\partial I}{\partial a^\mu}S^{\mu\nu}\frac{\partial H}{\partial a^\nu}. \quad (18)$$

Then it turns to solving the partial differential equation (17) to find the integral of Eqs.(2).

5. Illustrative example

In the following, an example is given to illustrate the application of the methods.

Solve the Bernoulli equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}-4y=x^2y^{1/2}. \quad (19)$$

Firstly, apply the Hamilton–Noether method in this example. Introducing an intermediate variable t, Eq.(19) can be expressed as

$$\dot{x} = x, \quad \dot{y} = 4y + x^2 y^{1/2}. \tag{20}$$

Setting $q = x$ and $p = y$, the expression (5) gives

$$\begin{aligned} H =& p\int_0^1 q\mathrm{d}\tau = pq, \\ Q =& \frac{\partial H}{\partial q} + 4p + q^2 p^{1/2}. \end{aligned}$$

The Noether identity (7) yields

$$p\dot{\xi} + (4p + q^2 p^{1/2})(\xi - \dot{q}\xi_0) - p(q\dot{\xi}_0 + \dot{q}\xi_0) + \dot{G}_N = 0.$$

It has the following solution:

$$\xi_0 = 0, \quad \xi = \frac{1}{2}p^{-1/2}q^{-2}, \quad G_N = \frac{1}{2}p^{1/2}q^{-2} - \frac{1}{2}\ln q.$$

The Noether conserved quantity expression (8) gives

$$I_N = p^{1/2}q^{-2} - \frac{1}{2}\ln q = \text{const.}$$

Rewriting the above expression in coordinates x and y, we have

$$I_N = y^{1/2}x^{-2} - \frac{1}{2}\ln x = \text{const.} \tag{21}$$

Secondly, apply the Lagrange–Noether method in this example. Eliminating x from Eqs.(20), we obtain

$$\ddot{y} = 4\dot{y} + \frac{1}{2}y^{-1}\dot{y}^2 - 8y. \tag{22}$$

It can be transformed into a holonomic nonconservative system; its Lagrangian and generalized force are

$$L = \frac{1}{2}\dot{y}^2 - 4y^2, \quad Q = 4\dot{y} + \frac{1}{2}y^{-1}\dot{y}^2,$$

respectively. The Noether identity (13) gives

$$\begin{aligned} &L\dot{\xi}_0 + \dot{y}(\dot{\xi} - \dot{y}\dot{\xi}_0) - 8y\xi \\ &+ \left(4\dot{y} + \frac{1}{2}y^{-1}\dot{y}^2\right)(\xi - \dot{y}\xi_0) + \dot{G}_N = 0. \end{aligned}$$

It has a solution

$$\begin{aligned} \xi_0 =& 0, \quad \xi = -\frac{1}{4}\dot{y}(\dot{y} - 4y)^{-2}, \\ G_N =& y(\dot{y} - 4y)^{-1} - \frac{1}{4}\ln\left[y^{-1/2}(\dot{y} - 4y)\right] \\ &+ \frac{1}{4}\dot{y}^2(\dot{y} - 4y)^{-2}. \end{aligned}$$

The Noether conserved quantity expression (14) yields

$$I_N = y(\dot{y} - 4y)^{-1} - \frac{1}{4}\ln\left[y^{-1/2}(\dot{y} - 4y)\right] = \text{const.} \tag{23}$$

Substituting the second equation of Eqs.(20) into the above expression, we obtain

$$I_N = y^{1/2}x^{-2} - \frac{1}{2}\ln x = \text{const.} \tag{24}$$

Finally, apply the Poisson method in this example. Setting

$$a^1 = x, \quad a^2 = y,$$

the generalized Poisson condition (17) gives

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^1}a^1 + \frac{\partial I}{\partial a^2}\left[4a^2 + (a^1)^2(a^2)^{1/2}\right] = 0.$$

It is equivalent to the following equation:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}a^1}{a^1} = \frac{\mathrm{d}a^2}{4a^2 + (a^1)^2(a^2)^{1/2}},$$

from which we obtain

$$\mathrm{d}a^1 = a^1\mathrm{d}t, \quad \mathrm{d}a^2 = \left[4a^2 + (a^1)^2(a^2)^{1/2}\right]\mathrm{d}t.$$

Then we have

$$\begin{aligned} &-\left[2(a^1)^{-3}(a^2)^{1/2} + \frac{1}{2}(a^1)^{-1}\right]\mathrm{d}a^1 \\ &+ \frac{1}{2}(a^1)^{-2}(a^2)^{-1/2}\mathrm{d}a^2 \\ =& -\left[2(a^1)^{-3}(a^2)^{1/2} + \frac{1}{2}(a^1)^{-1}\right]a^1\mathrm{d}t \\ &+ \frac{1}{2}(a^1)^{-2}(a^2)^{-1/2}\left[4a^2 + (a^1)^2(a^2)^{1/2}\right]\mathrm{d}t \\ =&0. \end{aligned}$$

Integrating the above expression, we obtain

$$(a^1)^{-2}(a^2)^{1/2} - \frac{1}{2}\ln a^1 = \text{const.} \tag{25}$$

Rewriting the above expression in coordinates x and y, we have

$$y^{1/2}x^{-2} - \frac{1}{2}\ln x = \text{const.} \tag{26}$$

6. Summary

The classical analytical mechanics, which once was a branch of applied mathematics, needs to use a lot of mathematical methods. On the contrary, some results of analytical mechanics can be applied in mathematics. This paper shows how to use three methods of analytical mechanics to solve a differential equation of first order.

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