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The homotopic mapping method for sea–air oscillator model of interdecadal climate fluctuations*

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The El Niño/La Niña Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. In this paper, a coupled system of sea–air oscillator model is studied. The aim is to create an asymptotic solving method of nonlinear equation for the ENSO model. And based on a class of oscillators of ENSO model, employing the method of homotopic mapping, the approximate solution of corresponding problem is studied. It is proven from the results that the homotopic method can be used for analysing the sea surface temperature anomaly in the equatorial eastern Pacific and the thermocline depth anomaly of the atmosphere–ocean oscillation for ENSO model.

Keywords: nonlinear, homotopic mapping, El Niño-Southern oscillator model
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1. Introduction

The El Niño Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interactions. The phenomenon of the ENSO is a very attractive object of study in international academic circles.[1−7] Mo and Lin *et al* also considered a class of problems in atmospheric physics, ocean weather, dynamic system and so on.[8−26] Superimposed on this natural mode of the coupled ocean–atmosphere system is the interdecadal fluctuation that contributes to the irregularity of the Southern Oscillation. In the eastern equatorial tropical Pacific a shoaling of subsurface isotherms signals the end of El Niño and the return of colder surface waters. The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. This anomaly reduces the zonal sea surface temperature (SST) gradient and hence the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation and changes that reinforce SST anomaly. This positive ocean–atmosphere feedback or coupled ocean–atmosphere instability leads the equatorial Pacific to a never-ending warm state.

Recently, many scholars have investigated the nonlinear problem.[27−33] Approximate methods have been developed and refined, including the method of averaging, boundary layer method, matched asymptotic expansion method and multiple scales method. In this paper we study a sea–air oscillator model using a simple and valid method for homotopic mapping theory.

2. Sea–air oscillator model

The response of the atmosphere to the warming in the tropics involves an intensification of the extratropical westerly, leading to colder surface waters in

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extratropical regions that happen to be windows to the equatorial thermocline. It implies a continual interdecadal climate fluctuation with a period which it takes for water parcels to travel from the extratropics to the equator.

Now we consider the coupled system for a sea–air oscillator model of interdecadal climate fluctuations:[2]

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \alpha\gamma(T_2 - T_1) + \lambda_1 T_2 - \varepsilon T_1^3 + Q^*, \quad (1)$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = -\gamma(T_2 - T_1) - \delta(T_2 - T_1(t-d)) + \lambda_2 T_2 - \varepsilon T_2^3, \quad (2)$$

where T_1 is the extratropical temperature at an approximate region 25°N to 50°N (and 25°S to 50°S), T_2 is the tropical temperature at an approximate region 20°S to 20°N, α is the fraction of poleward atmospheric heat transport, λ_1 is a negative constant, λ_2 is a feedback parameter, γ, δ, ε are positive constants, d is the delay time and Q^* represents the stochastic forcing from weather systems unrelated to tropical temperature variations. In this paper, we assume that $Q^* = 0$.

3. Homotopic mapping

Introducing a set of homotopic mapping[34] $H_i(T_1, T_2, p): X \times Y \times I \to R, i = 1, 2$:

$$\begin{aligned} H_1(T_1, T_2, p) =& L_1(T_1, T_2) - L_1(u_1, u_2) \\ &+ p(L_1(u_1, u_2) + \varepsilon(T_1)^3), \end{aligned} \quad (3)$$

$$\begin{aligned} H_2(T_1, T_2, p) =& L_2(T_1, T_2) - L_2(u_1, u_2) \\ &+ p(L_2(u_1, u_2) - \delta T_1(t-d)) + \varepsilon T_2^3), \end{aligned} \quad (4)$$

where $X = [0, \infty)$, $Y = [0, \infty)$, $I = [0, 1]$, $R = (-\infty, +\infty)$, while linear operators L_i $(i = 1, 2)$ are

$$L_1(T_1, T_2) = \frac{\mathrm{d}T_1}{\mathrm{d}t} + \alpha\gamma T_1 - (\alpha\gamma + \lambda_1)T_2, \quad (5)$$

$$L_2(T_1, T_2) = \frac{\mathrm{d}T_2}{\mathrm{d}t} - \gamma T_1 + (\gamma + \delta - \lambda_2)T_2, \quad (6)$$

and (u_1, u_2) is an initial approximation of solution for the original system (1), (2).

From Eqs.(5), (6), let

$$L_1(u_1, u_2) = 0, \quad (7)$$

$$L_2(u_1, u_2) = 0. \quad (8)$$

We have

$$u_1(t) = C_{10}\exp(r_1 t) + C_{20}\exp(r_2 t), \quad (9)$$

$$u_2(t) = D_{10}\exp(r_1 t) + D_{20}\exp(r_2 t)], \quad (10)$$

where C_{i0} $(i = 1, 2)$ are arbitrary constants, $D_{i0} = [(r_i + \alpha\gamma)C_{i0}/(\alpha\gamma + \lambda_1)$ $(i = 1, 2)$ and r_i $(i = 1, 2)$ are characteristic roots of Eqs.(7), (8):

$$r_{1,2} = -\frac{1}{2}(\alpha\gamma + \gamma + \delta - \lambda_2) \pm \frac{1}{2}\sqrt{\Delta},$$

$$\Delta = (\alpha\gamma + \gamma + \delta - \lambda_2)^2 + 4\gamma(\alpha\gamma + \lambda_1 - \alpha\delta).$$

Obviously, from Eqs.(3), (4), $H_j(T_1, T_2, 1) = 0$ $(j = 1, 2)$ are the same as the system (1), (2). Then the solution $(T_1(t), T_2(t))$ to the coupled system (1), (2) is the same as the solution of $H_j(T_1, T_2, p) = 0$ $(j = 1, 2)$ as $p \to 1$.

4. Approximate solution

Let

$$T_1 = \sum_{k=0}^{\infty} (T_1(t))_k p^k, \quad (11)$$

$$T_2 = \sum_{k=0}^{\infty} (T_2(t))_k p^k. \quad (12)$$

Considering Eqs.(11), (12) and Eqs.(3), (4), developed to the power series of p, we compare the coefficients of the same power in p for two sides of the equations.

From the coefficients of zeroth power in p, we have

$$L_i((T_1)_0, (T_2)_0) = L_i(u_0, v_0), \quad i = 1, 2.$$

Obviously, then, we have

$$(T_1(t))_0 = C_{10}\exp(r_1 t) + C_{20}\exp(r_2 t), \quad (13)$$

$$(T_2(t))_0 = D_{10}\exp(r_1 t) + D_{20}\exp(r_2 t). \quad (14)$$

From Eqs.(3), (4) and the coefficients of first power in p, we obtain

$$\begin{aligned} &L_1((T_1)_1, (T_2)_1) \\ =& -\varepsilon(C_{10}\exp(r_1 t) + C_{20}\exp(r_2 t))^3, \end{aligned} \quad (15)$$

$$\begin{aligned} &L_2((T_1)_1, (T_2)_1) \\ =& \delta\exp(-d)(C_{10}\exp(r_1 t) + C_{20}\exp(r_2 t)) \\ &- \varepsilon(D_{10}\exp(r_1 t) + D_{20}\exp(r_2 t))^3. \end{aligned} \quad (16)$$

From Eqs.(15), (16), we obtain

$$\begin{aligned} (T_1(t))_1 =& C_{11}\exp(r_1 t) + C_{21}\exp(r_2 t) \\ &+ A_{11}\exp(3r_1 t) + A_{21}\exp((2r_1 + r_2)t) \\ &+ A_{31}\exp((r_1 + 2r_2)t) + A_{41}\exp(3r_2 t), \end{aligned} \quad (17)$$

$$\begin{aligned} (T_2(t))_1 =& D_{11}\exp(r_1 t) + D_{21}\exp(r_2 t) \\ &+ \delta\exp(-d)t(C_{10}\exp(r_1 t) + C_{20}\exp(r_2 t)) \\ &+ B_{11}\exp(3r_1 t) + B_{21}\exp((2r_1 + r_2)t) \\ &+ B_{31}\exp((r_1 + 2r_2)t) + B_{41}\exp(3r_2 t), \end{aligned} \quad (18)$$

where C_{i1} $(i = 1, 2)$ are arbitrary constants; $D_{i1} = (r_i + \alpha\gamma)C_{i1}/(\alpha\gamma + \lambda_1)$ $(i = 1, 2)$ and

$$\begin{aligned}
A_{11} =& -\frac{\varepsilon}{\Delta_1}[C_{10}^3(3r_1 + \gamma + \delta - \lambda_2) \\
&+ D_{10}^3(\alpha\gamma + \lambda_2)], \\
A_{21} =& -\frac{3C_{10}D_{10}\varepsilon}{\Delta_2}[C_{10}(2r_1 + r_2 + \gamma + \delta - \lambda_2) \\
&+ D_{10}(\alpha\gamma + \lambda_2)], \\
A_{31} =& -\frac{3C_{10}D_{10}\varepsilon}{\Delta_3}[D_{10}(r_1 + 2r_2 + \gamma + \delta - \lambda_2) \\
&+ C_{10}(\alpha\gamma + \lambda_2)] \\
A_{41} =& -\frac{\varepsilon}{\Delta_4}[C_{10}^3(3r_2 + \gamma + \delta - \lambda_2) \\
&+ D_{10}^3(\alpha\gamma + \lambda_2)], \\
B_{11} =& -\frac{\varepsilon}{\Delta_1}[C_{10}^3\gamma + D_{10}^3(3r_1 + \alpha\gamma)], \\
B_{21} =& -\frac{3C_{10}D_{10}\varepsilon}{\Delta_2}[C_{10}\gamma + D_{10}(2r_1 + r_2 + \alpha\gamma)], \\
B_{31} =& -\frac{3C_{10}D_{10}\varepsilon}{\Delta_3}[C_{10}\gamma + D_{10}(r_1 + 2r_2 + \alpha\gamma)], \\
B_{41} =& -\frac{\varepsilon}{\Delta_4}[C_{10}^3\gamma + D_{10}^3(3r_2 + \alpha\gamma)],
\end{aligned}$$

$$\Delta_1 = \begin{vmatrix} 3r_1 + \alpha\gamma & -(\alpha\gamma + \lambda_1) \\ -\gamma & 3r_1 + \gamma + \delta - \lambda_2 \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} 2r_1 + r_2 + \alpha\gamma & -(\alpha\gamma + \lambda_1) \\ -\gamma & 2r_1 + r_2 + \gamma + \delta - \lambda_2 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} r_1 + 2r_2 + \alpha\gamma & -(\alpha\gamma + \lambda_1) \\ -\gamma & r_1 + 2r_2 + \gamma + \delta - \lambda_2 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} 3r_2 + \alpha\gamma & -(\alpha\gamma + \lambda_1) \\ -\gamma & 3r_2 + \gamma + \delta - \lambda_2 \end{vmatrix}.$$

Thus we can obtain the first order approximate solution of the system (1), (2):

$$T_{1app}(t) = (T_1(t))_0 + (T_1(t))_1, \tag{19}$$

$$T_{2app}(t) = (T_2(t))_0 + (T_2(t))_1. \tag{20}$$

where $(T_1(t))_i$, $(T_2(t))_i$ $(i = 0, 1)$ are determined by Eqs.(13), (14), (17) and (18), respectively.

Using the same method, we can obtain higher-order approximate solutions to the coupled system (1), (2).

5. Discussion

1. From the constructions of Eqs.(1), (2) and homotopic mapping (3), (4), we can prove that expressions (11), (12) of the extratropical temperature T_1 and tropical temperature T_2 are convergent series in $p \in [0, 1]$.[34] And then expressions (11), (12) are a set of solution to the coupled system (1), (2) of a sea–air oscillator model of interdecadal climate fluctuations.

2. From homotopic mapping (3), (4), it is of importance that we should select a series of initial approximations of solution (u_1, u_2) for the original system (1), (2). We can also prove that as a series solution to the system (7), (8) there is a faster convergent precision. Thus the first-order approximations (19), (20) of solution to the system (1), (2) are better approximate solutions.

3. Atmospheric physics is a science studying very complicated natural phenomena. Hence we need to establish reduced basic models for the sea–air oscillator. And then we solve them using approximate methods. The homotopic mapping method is a simple and valid method.

4. The homotopic mapping method is an approximate analytic method, which differs from the general numerical method. The expansion of the solution using the homotopic mapping method can be continuous analytic operation. From Eqs.(19), (20), we can further study the qualitative and quantitative behaviours of the temperature anomaly in the equatorial eastern Pacific, the thermocline anomaly and the zonal wind stress anomaly.

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