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Grover search algorithm in an ion trap system*

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Two schemes for the implementation of the two-qubit Grover search algorithm in the ion trap system are proposed. These schemes might be experimentally realizable with presently available techniques. The experimental implementation of the schemes would be an important step toward more complex quantum computation in the ion trap system.

Keywords: grover search algorithm, ion trap system, quantum computation
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Quantum computers, which are based on the fundamental quantum mechanical principle, can solve some problems exponentially faster than the classical computers.[1] For example, Shor[2] has shown that the problem of factorizing a large integer, the basis of the security of many cryptographic systems, can be solved in polynomial time by using a quantum computer, while it takes exponentially increasing time to solve on a classical computer. On the other hand, Grover[3] has shown that a search with a quantum computer for an item from a disordered system needs much shorter time than that with a classical computer.

We first give a brief review on the Grover's search algorithm. The goal is to find one item among N items stored in an unsorted database. The database can be accessed by a "black box" oracle comparing any item with the searched one. It gives the "Yes" answer when the item matches, "No" otherwise. The most efficient classical algorithm is to examine items one by one until the oracle returns "Yes". On average, $N/2$ inquiries are necessary. In the quantum search algorithm, multiple items are simultaneously examined by using a superposition of the corresponding states. The quantum search for the marked item requires only $O(\sqrt{N})$ inquiries.

The database items are represented by a quantum resister with n qubits, having 2^n possible states, $|0\rangle = |00...0\rangle$, $|1\rangle = |00...1\rangle$, $|2^{n-1}\rangle = |11...1\rangle$. Let us assume that the marked item corresponds to the state $|m\rangle$ and that the Yes/No answer from the oracle is encoded in a $\pi/0$ phase shift. The transformation performed by the oracle is thus $I_m = I - 2|m\rangle\langle m|$, where I is the $2^n \times 2^n$ identity matrix on the quantum resister. Note that I_m amounts to a conditional phase operation. The algorithm consists of a repetition of the transformation

$$Q = HI_0HI_m, \tag{1}$$

where $I_0 = I - 2|0\rangle\langle 0|$ and $H = \prod_{i=1}^{n} H_i$, with H_i being the Hadmard gate on the corresponding qubit

$$\begin{aligned} |0\rangle_i &\longrightarrow (|0\rangle_i + |1\rangle_i)/\sqrt{2}, \\ |1\rangle_i &\longrightarrow (|0\rangle_i - |1\rangle_i)/\sqrt{2}. \end{aligned} \tag{2}$$

The sequence of Q operations acts on a state prepared initially in $|\Psi\rangle = \frac{1}{\sqrt{N}} \prod_{i=1}^{n} |i\rangle$. All states are then examined by the oracle. After the operation I_m, only the state $|m\rangle$ gains minus sign and the resulting resister state is $\frac{1}{\sqrt{N}} (\prod_{i\neq m} |i\rangle - |m\rangle)$. The operation HI_0H finally performs an inversion-about-the-average operation, which increases the probability amplitude of $|m\rangle$. After $O(\sqrt{N})$ iterations of the elementary transformation, the probability to get the resister in the state $|m\rangle$ is maximum and of the order of unity. Thus, a simple read-out of the resister provides the searched item.

In the simple case of two qubits, on which we will focus from now on, there are four items: $|0\rangle = |00\rangle$, $|1\rangle = |01\rangle$, $|2\rangle = |10\rangle$, and $|3\rangle = |11\rangle$. After I_m, the amplitude of the four states is $1/2$ for $i \neq m$ and $-1/2$ for $i = m$. After HI_0H the amplitude becomes 0 for $i \neq m$ and 1 for $i = m$. Grover's search thus requires

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only one transformation Q and one inquiry of the oracle. The H_i transformations are single-qubit gates, which are easily performed. The most critical part of the algorithm is the realization of I_0 and I_m, which produce an entangled state of the two qubits. We note that I_0 is essentially the quantum phase gate, performing a conditional phase shift on the resister only when it is in the state $|0\rangle$. For $m = 0$, I_m is the phase gate I_0. For $m \neq 0$, I_m can be obtained by I_0 plus a single qubit transformation.

Recently, a scheme has been proposed for realizing the Grover search algorithm in cavity QED;[4] however, the experimental demonstration has not been reported yet. On the other hand, recent advance in ion trap has opened prospects for quantum information processing. In recent experiments involving two or three trapped ions,[5−7] the control gate and entanglement operations between different ions have been achieved. However, only the simplest Deutsch–Jozsa algorithm has been implemented in such a system.[8] In this paper we propose two schemes for the implementation of the two-qubit Grover search algorithm in ion trap system. Our schemes might be experimentally realizable with presently available techniques.[5−8] The experimental implementation of our schemes would be an important step toward more complex quantum algorithm, serving to illustrate the power of the ion trap system for quantum information processing.

We consider two identical three-level ions, which have two ground states $|e\rangle$ and $|g\rangle$ and an excited state $|r\rangle$ confined in a linear trap. The transition $|e\rangle \rightarrow |r\rangle$ is driven by two classical laser fields with detunings Δ and $\Delta + \omega_1 - \delta$, with ω_1 being the frequency of the centre-of-mass vibrational mode and δ being the difference between ω_1 and the relative detuning of the two lasers. The first laser is perpendicular to the trap axis and thus does not excite the vibrational modes along the trap axis, while the second laser propagates along the trap axis. The relative detuning of the two lasers is close to ω_1: i.e., $\omega_1 \gg \delta$. In this case, we can neglect other vibrational modes. In the rotating-wave approximation, the Hamiltonian for this system is given by (assuming $\hbar = 1$)[9,10]

$$H = \omega_1 a^+ a + \sum_{j=1}^{2} \left[\frac{\Omega_1}{2} |r_j\rangle \langle e_j| \, \mathrm{e}^{-\mathrm{i}[\Delta t - \eta(a+a^+)]} + \frac{\Omega_2}{2} |r_j\rangle \langle e_j| \, \mathrm{e}^{-\mathrm{i}[(\Delta+\omega_1-\delta)t - \eta(a+a^+)]} + \mathrm{H.c.} \right], \quad (3)$$

where a^+ and a are the creation and annihilation operators of the vibrational mode respectively, Ω_j (j=1, 2) is the Rabi frequency of the jth laser, and $\eta = k_2/\sqrt{2\omega_1 M}$ is the Lamb–Dicke parameter, with k_2 being the wavevector of the second laser and M being the mass of the two ions. Assume that the lasers are far off-resonant with the ions, i.e., $\Delta \gg \Omega_j, \omega_1$. In this case, the ions do not undergo any real transition. According to the perturbation theory the lasers induce energy shift. The energy shift for the state $|r_j\rangle$ is

$$\frac{\langle r_j| H |e_j\rangle \langle e_j| H |r_j\rangle}{\Delta} = \frac{1}{2} \left\{ \frac{\Omega_1^2 + \Omega_2^2}{2\Delta} + \Omega \mathrm{e}^{-\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} + \Omega \mathrm{e}^{\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} \right\}, \quad (4)$$

where

$$\Omega = \frac{\Omega_1 \Omega_2}{2\Delta}. \quad (5)$$

The first term in Eq.(4) describes the process that each ion may absorb one photon from one laser and then immediately emit one photon to the same laser. The other two terms describe the process that each ion may absorb one photon from one laser and then immediately emit one photon to the other laser. On the other hand, the energy shift for the state $|e_j\rangle$ is

$$\frac{\langle e_j| H |r_j\rangle \langle r_j| H |e_j\rangle}{-\Delta} = -\frac{1}{2} \left\{ \frac{\Omega_1^2 + \Omega_2^2}{2\Delta} + \Omega \mathrm{e}^{-\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} + \Omega \mathrm{e}^{\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} \right\}. \quad (6)$$

Thus, the effective Hamiltonian is

$$H_\mathrm{e} = \omega_1 a^+ a + \sum_{j=1}^{2} \left\{ \frac{\Omega_1^2 + \Omega_2^2}{2\Delta} + \Omega \mathrm{e}^{-\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} + \Omega \mathrm{e}^{\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} \right\} \sigma_{z,j}, \quad (7)$$

where

$$\sigma_{z,j} = \frac{1}{2}(|r_j\rangle \langle r_j| - |e_j\rangle \langle e_j|). \quad (8)$$

If the ions initially have no probabilities of being populated in the state $|r\rangle$, they will remain in the ground states during the interaction. In this case the Hamiltonian reduces to

$$H_\mathrm{e} = \omega_1 a^+ a - \sum_{j=1}^{2} \left\{ \frac{\Omega_1^2 + \Omega_2^2}{4\Delta} + \frac{\Omega}{2} \mathrm{e}^{-\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} + \frac{\Omega}{2} \mathrm{e}^{\mathrm{i}[(\omega_1-\delta)t - \eta(a+a^+)]} \right\} |e_j\rangle \langle e_j|. \quad (9)$$

Consider the behaviour of the trapped ions in the Lamb–Dicke regime, where $\eta\sqrt{\bar{n}+1} \ll 1$, with $\bar{n}$ being the mean phonon number of the centre-of-mass mode. We can expand the Hamiltonian to the first order in η

$$\begin{aligned} H_{\rm e} =&\omega_1 a^+ a - \sum_{j=1}^{2}\left[\frac{\Omega_1^2+\Omega_2^2}{4\Delta} + \Omega\cos(\omega_1-\delta)t \right. \\ &+ {\rm i}\eta\frac{\Omega}{2}{\rm e}^{-{\rm i}(\omega_1-\delta)t}(a+a^+) \\ &\left. - {\rm i}\eta\frac{\Omega}{2}{\rm e}^{{\rm i}(\omega_1-\delta)t}(a+a^+)\right]|e_j\rangle\langle e_j| . \end{aligned} \tag{10}$$

In the interaction picture the interaction Hamiltonian is given by

$$\begin{aligned} H_i =& -\sum_{j=1}^{2}\left\{\frac{\Omega_1^2+\Omega_2^2}{4\Delta} + \Omega\cos(\omega_1-\delta)t \right. \\ &+ {\rm i}\eta\frac{\Omega}{2}a^+[{\rm e}^{{\rm i}\delta t} - {\rm e}^{{\rm i}(2\omega_1-\delta)t}] \\ &\left. - {\rm i}\eta\frac{\Omega}{2}a[{\rm e}^{-{\rm i}\delta t} - {\rm e}^{-{\rm i}(2\omega_1-\delta)t}]\right\}|e_j\rangle\langle e_j| . \end{aligned} \tag{11}$$

Assume that $\eta\Omega \ll 2\omega_1 - \delta$. In this case the terms $-{\rm i}\eta\frac{\Omega}{2}a^+{\rm e}^{{\rm i}[(2\omega_1-\delta)t+\phi]} + {\rm H.c.}$ oscillate very fast and thus can be neglected.[9,11] Then the interaction Hamiltonian reduces to

$$\begin{aligned} H_i =& -\sum_{j=1}^{2}\left\{\frac{\Omega_1^2+\Omega_2^2}{4\Delta} + \Omega\cos(\omega_1-\delta)t \right. \\ &\left. + {\rm i}\eta\frac{\Omega}{2}(a^+{\rm e}^{{\rm i}\delta t} - a{\rm e}^{-{\rm i}\delta t})\right\}|e_j\rangle\langle e_j| . \end{aligned} \tag{12}$$

The evolution of the system is[5,12]

$$\begin{aligned} &|g_1\rangle|g_2\rangle|\phi_v(0)\rangle \longrightarrow |g_1\rangle|g_2\rangle|\phi_v(0)\rangle \\ &|e_1\rangle|g_2\rangle|\phi_v(0)\rangle \longrightarrow {\rm e}^{{\rm i}\theta}{\rm e}^{{\rm i}\phi}D(\alpha)|e_1\rangle|g_2\rangle|\phi_v(0)\rangle , \\ &|g_1\rangle|e_2\rangle|\phi_v(0)\rangle \longrightarrow {\rm e}^{{\rm i}\theta}{\rm e}^{{\rm i}\phi}D(\alpha)|g_1\rangle|e_2\rangle|\phi_v(0)\rangle , \\ &|e_1\rangle|e_2\rangle|\phi_v(0)\rangle \longrightarrow {\rm e}^{{\rm i}2\theta}{\rm e}^{4i\phi}D(2\alpha)|g_1\rangle|e_2\rangle|\phi_v(0)\rangle , \end{aligned} \tag{13}$$

where

$$\alpha = -\int_0^\tau \eta\frac{\Omega}{2}{\rm e}^{{\rm i}\delta t'}{\rm d}t' = -\eta\frac{\Omega}{2{\rm i}\delta}({\rm e}^{{\rm i}\delta\tau}-1), \tag{14}$$

$$D(\alpha) = {\rm e}^{\alpha a^+ - \alpha^* a}, \tag{15}$$

$$\begin{aligned} \phi =& I_m\int_\gamma \alpha'^*{\rm d}\alpha' \\ =& -I_m\int_0^\tau\left(\eta\frac{\Omega}{2}\right)^2\frac{1}{{\rm i}\delta}(1-{\rm e}^{{\rm i}\delta t'}){\rm d}t' \\ =& \left(\eta\frac{\Omega}{2}\right)^2\left[\frac{\tau}{\delta} - \frac{1}{\delta^2}\sin\delta\tau\right], \end{aligned} \tag{16}$$

$$\alpha' = -\eta\frac{\Omega}{2{\rm i}\delta}({\rm e}^{{\rm i}\delta t'}-1), \tag{17}$$

$$\theta = \frac{\Omega_1^2+\Omega_2^2}{4\Delta}\tau + \frac{\Omega}{\omega_1-\delta}\sin(\omega_1-\delta)\tau. \tag{18}$$

$|\phi_v(0)\rangle$ is the initial state of the vibrational mode. With the choice $\delta\tau = 2\pi$, the vibrational mode returns to the original point in the phase space, i. e., $\alpha = 0$, and ϕ reduces to

$$\phi = \left(\eta\frac{\Omega}{2}\right)^2\frac{2\pi}{\delta^2}, \tag{19}$$

ϕ is controllable by the Rabi frequencies of the laser fields. Then we obtain the evolution of the internal state

$$\begin{aligned} &|g_1\rangle|g_2\rangle \longrightarrow |g_1\rangle|g_2\rangle \\ &|e_1\rangle|g_2\rangle \longrightarrow {\rm e}^{{\rm i}\theta}{\rm e}^{{\rm i}\phi}|e_1\rangle|g_2\rangle , \\ &|g_1\rangle|e_2\rangle \longrightarrow {\rm e}^{{\rm i}\theta}{\rm e}^{{\rm i}\phi}|g_1\rangle|e_2\rangle , \\ &|e_1\rangle|e_2\rangle \longrightarrow {\rm e}^{{\rm i}2\theta}{\rm e}^{4i\phi}|e_1\rangle|e_2\rangle . \end{aligned} \tag{20}$$

We now assume that the quantum logic states $|1\rangle$ and $|0\rangle$ are represented by $|g\rangle$ and $|e\rangle$ respectively. Both ions are first prepared in the state $(|e\rangle+|g\rangle)/\sqrt{2}$. Then I_0 is achieved by this transformation plus the single-qubit transformation: $|e_{1,2}\rangle \longrightarrow {\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_{1,2}\rangle$ with the choice $\phi = \pi/2$. For $m = 1$, I_m is obtained by the transformation plus the single-qubit transformation: $|e_1\rangle \longrightarrow -{\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_1\rangle$ and $|e_2\rangle \longrightarrow {\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_2\rangle$. For $m = 2$, 3, I_m is obtained by the transformation plus the single-qubit transformations: $|e_1\rangle \longrightarrow {\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_1\rangle$ and $|e_2\rangle \longrightarrow -{\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_2\rangle$; $|e_{1,2}\rangle \longrightarrow {\rm i}{\rm e}^{-{\rm i}\theta}{\rm e}^{-{\rm i}\phi}|e_{1,2}\rangle$ and $|g_{1,2}\rangle \longrightarrow -{\rm i}|g_{1,2}\rangle$, respectively.

We note that the Grover's search algorithm can also be realized in an ion trap by using the vibrational mode as the data bus. We here assume that the ions have two excited metastable states $|e\rangle$ and $|e'\rangle$ and one ground state $|g\rangle$. The quantum bit (qubit) is carried by states $|e\rangle$ and $|g\rangle$. Both ions are first prepared in the state $(|e\rangle + |g\rangle)/\sqrt{2}$. We now drive the first ion with a laser tuned to the first upper vibrational sideband with respect to the transition $|g\rangle \rightarrow |e\rangle$. Assume that the laser is off-resonant with the transition $|g\rangle \rightarrow |e'\rangle$ and thus the state $|e'\rangle$ is not affected during the interaction. In the Lamb–Dicke limit and the weak-excitation regime where the Rabi frequency is much smaller than the vibrational frequency, the interaction Hamiltonian is[13]

$$H = {\rm i}\eta\Omega{\rm e}^{-{\rm i}\phi}a^+|e_1\rangle\langle g_1| + {\rm H.c.}, \tag{21}$$

where a^+ and a are the creation and annihilation operators of the centre-of-mass vibrational mode of the trapped ions, η is the Lamb–Dicke parameter, and Ω and ϕ are the Rabi frequency and phase of this laser field. After an interaction time τ_1 the state of the system combined by this ion and the centre-of-mass mode is

$$\sqrt{\frac{1}{2}}\,|e_1\rangle\,|0\rangle + \sqrt{\frac{1}{2}}[\cos(\eta\Omega\tau)\,|g_1\rangle\,|0\rangle + \mathrm{e}^{-\mathrm{i}\phi}\sin(\eta\Omega\tau)\,|e_1\rangle\,|1\rangle]. \tag{22}$$

With the choice $\phi = 0$ and $\eta\Omega\tau_1 = \pi/2$ we obtain

$$|e_1\rangle\,(|0\rangle + |1\rangle)/\sqrt{2}. \tag{23}$$

By this way the electronic state of the first ion is encoded to the centre-of-mass motional mode.

We then drive the second ion with a laser tuned to the first lower vibrational sideband with respect to the transition $|g\rangle \to |\mathrm{e}'\rangle$. The interaction Hamiltonian is given by

$$H' = \mathrm{i}\eta\Omega'\mathrm{e}^{-\mathrm{i}\phi'}a\,|e_2'\rangle\,\langle g_2| + \mathrm{H.c.} \tag{24}$$

After an interaction time τ_2 the evolution of the state $|1\rangle\,|g_2\rangle$ is

$$|1\rangle\,|g_2\rangle \longrightarrow [\cos(\eta\Omega'\tau')\,|1\rangle\,|g_2\rangle - \mathrm{e}^{-\mathrm{i}\phi'}\sin(\eta\Omega'\tau')\,|0\rangle\,|e_2\rangle]. \tag{25}$$

On the other hand, $|0\rangle\,|e_2\rangle, |0\rangle\,|g_2\rangle, |1\rangle\,|e_2\rangle$ do not undergo any transition during the interaction. With the choice $\eta\Omega'\tau_2 = \pi$ we obtain $|1\rangle\,|g_2\rangle \longrightarrow -\,|1\rangle\,|g_2\rangle$. We now map the state of vibrational mode back to the first ion, leading to

$$\begin{aligned} |e_1\rangle\,|e_2\rangle &\longrightarrow |e_1\rangle\,|e_2\rangle \\ |e_1\rangle\,|g_2\rangle &\longrightarrow |e_1\rangle\,|g_2\rangle, \\ |g_1\rangle\,|e_2\rangle &\longrightarrow |g_1\rangle\,|e_2\rangle, \\ |g_1\rangle\,|g_2\rangle &\longrightarrow -\,|g_1\rangle\,|g_2\rangle. \end{aligned} \tag{26}$$

We here assume that the quantum logic states $|1\rangle$ and $|0\rangle$ are represented by $|e\rangle$ and $|g\rangle$ respectively. The above mentioned transformation corresponds to the control phase gate I_0. If $m \neq 0$, I_m can be obtained by I_0 plus the single qubit transformation Z: $|g\rangle_1 \longrightarrow -\,|g\rangle_1$ for $m = 1$; $|g\rangle_2 \longrightarrow -\,|g\rangle_2$ for $m = 2$; $|e\rangle_{1,2} \longrightarrow i\,|e\rangle_{1,2}$ and $|g\rangle_{1,2} \longrightarrow -\mathrm{i}\,|g\rangle_{1,2}$ for $m = 3$. Then we perform the Hadmard transformation on each qubit. After repeating the control phase gate I_0 and Hadmard transformation, we obtained the transformation for the Grover's search algorithm.

In conclusion, we have described two schemes for the implementation of the Grover's search algorithm with trapped ions. The required experimental techniques of the first scheme are within the scope of what can be obtained in the ion trap setup of the NIST group,[5,6] while the techniques of the second scheme are within the scope of what can be obtained in the setup of the Innsbruck group.[7,8] Therefore, based on presently available techniques we can say that our schemes might be realizable.

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