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Transition to complete synchronization via near-synchronization in two coupled chaotic neurons*

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The synchronization transition in two coupled chaotic Morris–Lecar (ML) neurons with gap junction is studied with the coupling strength increasing. The conditional Lyapunov exponents, along with the synchronization errors are calculated to diagnose synchronization of two coupled chaotic ML neurons. As a result, it is shown that the increase in the coupling strength leads to incoherence, then induces a transition process consisting of three different synchronization states in succession, namely, burst synchronization, near-synchronization and embedded burst synchronization, and achieves complete synchronization of two coupled neurons finally. These sequential transitions to synchronization reveal a new transition route from incoherence to complete synchronization in coupled systems with multi-time scales.

Keywords: synchronization transition, neuron coupling
PACC: 0545, 0547, 8725

1. Introduction

The synchronization of coupled oscillators was an interesting topic in nonlinear science in past decades. Chaotic synchronization has been studied extensively in the context of laser dynamics,[1] electric circuits,[2] chemical[3] and neural dynamics.[4−6] However, chaotic synchronization is still an attractive subject due to the inherent complexity of chaotic systems. For two coupled chaotic systems, several types of synchronization transitions were reported in Refs.[7,8]. Transitions of generalized synchronization and phase synchronization were also studied in Ref.[8]. It was shown that phase synchronization could appear before or after generalized synchronization. Phase synchronization in two coupled chaotic Hindmarsh–Rose (HR) neurons was studied by Shuai and Durand, who formulated the transition of phase synchronization and nearly complete synchronization by means of the Lyapunov exponent spectra, the phase error and the cross correlation function.[6] An internal relation between the onset of synchronization and the Lyapunov exponent spectra was investigated in detail in coupled multi-time scale systems.[9] Recently, the synchronization transition of chaotic bursts in neurons was reported in Ref.[10], in which a new transition style from burst synchronization to both spike and burst synchronizations was found in two coupled multi-time scale HR neurons.

Neurons often behave as collective coherent motions. The synchronization of coupled neurons has been suggested as a mechanism for binding globally distributing to a coherent motion and could play a key role in intercommunicating among neurons.[6] Synchronization behaviour is also of great importance for signal encoding and transduction in the information processing of neurons.[11] Due to the potential function of synchronization in collective behaviour of neurons, it is necessary to study synchronization transition further in mutually coupled neural systems.

In this work, the chaotic synchronization of two coupled ML neurons is investigated. The main goal is to explore the effect of the coupling strength on the process of transition to synchronization. It is found that at lower couplings the coupling might destroy the coherence in two coupled ML neurons. However,

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three different transition states in succession, that is, burst synchronization, near-synchronization and embedded burst synchronization, are induced to realize the complete synchronization by increasing the coupling further. Therefore, the change from incoherence to complete synchronization in this system is actually a complicated and versatile transition process.

This paper is organized as follows. A brief description of the ML neuron model is introduced in Section 2. In Section 3 the process of transition to synchronization is investigated in detail for two coupled chaotic ML neurons, and a conclusion is presented in Section 4.

2. Model description of Morris–Lecar neuron

The ML neuron model,[12] a model that is for electrical activity in the barnacle muscle fibre, is a simplified version of Hodgkin–Huxley neuron model for describing the firing and refractory properties of real neurons. This model contains a calcium current generating fast action potential and a rectifier potassium current. At the same time, a leakage current is considered to maintain a constant potential at the rest state. The reduced ML neuron model is given by the following differential equations:

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} =& g_{\mathrm{Ca}} m_\infty(V)(V_{\mathrm{Ca}} - V) \\ &+ g_k W(V_k - V) + g_l(V_l - V) - I, \\ \frac{\mathrm{d}W}{\mathrm{d}t} =& \lambda(V)(W_\infty(V) - W), \end{aligned} \tag{1}$$

with

$$\begin{aligned} m_\infty(V) &= 0.5\left(1 + \tanh\frac{V - V_{\mathrm{a}}}{V_{\mathrm{b}}}\right), \\ W_\infty(V) &= 0.5\left(1 + \tanh\frac{V - V_{\mathrm{c}}}{V_{\mathrm{d}}}\right), \\ \lambda(V) &= \frac{1}{3}\cosh\frac{V - V_{\mathrm{c}}}{2V_{\mathrm{d}}}, \end{aligned}$$

where t is the time variable, V the membrane action potential of the ML neuron model, W the probability of potassium(K^+) channel activation. In the present paper, the parameters used are given as follows: $g_{\mathrm{Ca}} = 1.2$, $g_k = 2.0$, $g_l = 0.5$, $V_k = -1.1$, $V_l = -0.5$, $V_{\mathrm{a}} = -0.01$, $V_{\mathrm{b}} = 0.15$, $V_{\mathrm{c}} = 0.1$, and $V_{\mathrm{d}} = 0.05$. Their detailed explanation is given in Ref.[12]. Among the parameters in Eq.(1), V_{Ca} is considered as a control parameter. To generate rich firing patterns in the ML neuron model, an external linear slow subsystem is introduced as follows:[13]

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \mu(0.2 + V), \tag{2}$$

where μ is a constant and we take $\mu = 0.005$ below. In this case, since μ is a small quantity, I is a slow variable, and V and W are the fast variables in the system (1), which is a double-time scale system now.

When the parameter V_{Ca} is changed, the ML neuron model exhibits the rich firing behaviour, such as various periodic and chaotic patterns. The bifurcation diagram of inter-spike interval (ISI) with respect to the parameter V_{Ca} is shown in Fig.1. To see various firings more clearly, we take V_{Ca}=0.950, 0.880, 0.845, and 0.800, respectively. Time evolutions of the corresponding membrane action potential are shown in Fig.2.

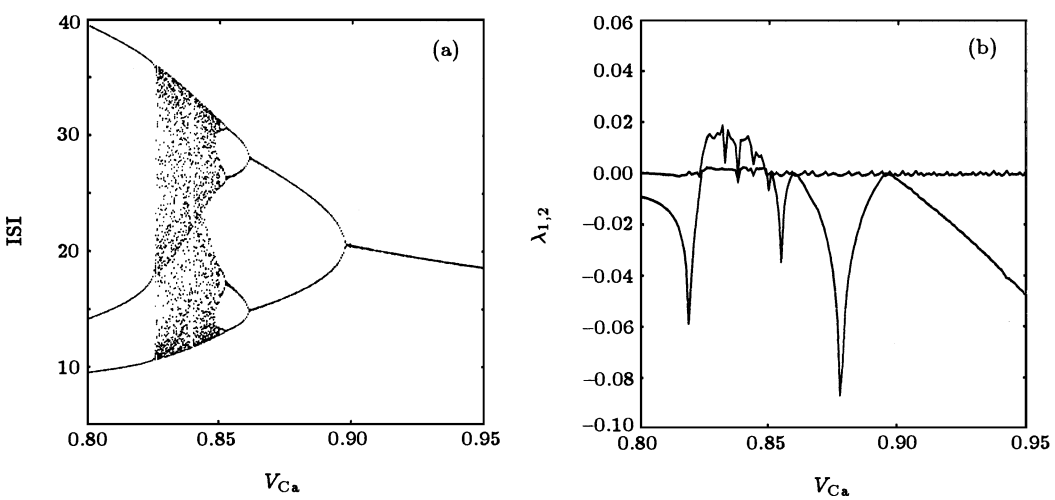


Fig.1. (a) Bifurcation diagram of ISI vs the parameter V_{Ca} in a single ML neuron. (b) Two largest Lyapunov exponents $\lambda_{1,2}$.

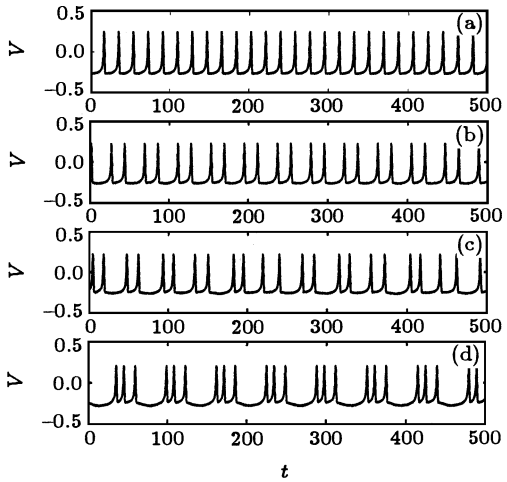


Fig.2. (a) Period-1 firing when $V_{\rm Ca} = 0.950$; (b) Period-2 firing when $V_{\rm Ca} = 0.880$; (c) Chaotic firing when $V_{\rm Ca} = 0.845$; (d) Period-3 firing when $V_{\rm Ca} = 0.800$.

3. Synchronization transition of two coupled chaotic ML neurons

In this section we consider two coupled ML neurons with gap junction as follows:

$$\begin{aligned}\frac{{\rm d}V_{1,2}}{{\rm d}t} =& g_{\rm Ca} m_\infty (V_{\rm Ca} - V_{1,2}) + g_k W (V_k - V_{1,2}) \\ &+ g_l (V_l - V_{1,2}) - I_{1,2} + C(V_{2,1} - V_{1,2}), \\ \frac{{\rm d}W_{1,2}}{{\rm d}t} =& \lambda(V_{1,2})(W_\infty(V_{1,2}) - W_{1,2}), \\ \frac{{\rm d}I_{1,2}}{{\rm d}t} =& \mu(0.2 + V_{1,2}), \end{aligned} \tag{3}$$

where C is the coupling strength of two coupled ML neurons with gap junction, and the subscript 1 (or 2) represents the neuron 1 (or 2). In the following study we take $V_{\rm Ca} = 0.845$ so that a single ML neuron exhibits chaotic firing behaviours as shown in Fig.2(c). To study the synchronization of two coupled chaotic neurons with gap junction, the synchronization errors are introduced by $e_1 = V_2 - V_1$, $e_2 = W_2 - W_1$, and $e_3 = I_2 - I_1$. The neurons reach complete synchronization if $|e_1|$, $|e_2|$ and $|e_3|$ vanish simultaneously in the limit of $t \longrightarrow +\infty$. It is obvious from the system (3) that $e_i (i = 1, 2, 3)$ satisfy the following linearized system:

$$\begin{aligned}\dot{e_1} =& \left(\left(-4 + 4\tanh\left(\frac{20}{3}V + \frac{1}{15}\right)^2 \right)(V - 0.845) \right. \\ &\left. - \frac{11}{10} - \frac{3}{5}\tanh\left(\frac{20}{3}V + \frac{1}{15}\right) - 2W - 2C \right) e_1 \\ &+ \left(-2V - \frac{11}{5} \right) e_2 - e_3, \\ \dot{e_2} =& \left(\frac{10}{3}\sinh(10V - 1) \left(\frac{1}{2} + \frac{1}{2}\tanh(20V - 2) - W \right) \right. \\ &\left. + \frac{1}{3}\cosh(10V - 1)(10 - 10\tanh(20V - 2)^2) \right) e_1 \\ &+ \left(-\frac{1}{3}\cosh(10V - 1) \right) e_2, \\ \dot{e_3} =& 0.005 e_1. \end{aligned} \tag{4}$$

To explore the stability of the synchronization manifold of the coupled systems with $V_1 = V_2$, $W_1 = W_2$ and $I_1 = I_2$, the conditional Lyapunov exponents (namely, the Lyapunov exponents of the linearized system (4)) are used. If all conditional Lyapunov exponents are negative, it is concluded that the stability of the synchronization manifold of two coupled systems is realized, which implies the occurrence of complete synchronization in two coupled neurons. We determine the conditional Lyapunov exponents of the linearized systems (4) by numerical methods.[14] The variations of two conditional Lyapunov exponents with the coupling strength increasing are shown in Fig.3. The bifurcation diagram of ISI of the first neuron in the coupled system vs the coupling strength C is given in Fig.4(a). It is seen from Fig.3 that the first conditional Lyapunov exponent remains positive until $C = 0.0885$. However, in the interval $0 < C < 0.0100$ with very low couplings, the second conditional Lyapunov exponent is negative. In more detail, the second conditional Lyapunov exponent decreases gradually in the interval $0 < C < 0.0060$, and the coupled neurons are in a nonsynchronous chaotic state. Then the second conditional Lyapunov exponent gradually increases in the interval $0.0060 < C < 0.0100$, in which two coupled neurons achieve a burst synchronization as illustrated in Fig.4(a).

It is seen clearly that both of the conditional Lyapunov exponents become positive in a larger interval $0.010 < C < 0.0630$, and this leads to incoherence as illustrated in Fig.4(a). The type and the size of the chaotic attractor vary rapidly and frequently. The main reason for coherence descending is that there exists a multi-scroll attractor in this multi-time scale ML neuron system as shown in Fig.4(b). As discussed similarly in Ref.[15], in a Lorenz system with a double-scroll attractor, when a trajectory inside this Lorenz attractor switches back and forth between two scrolls, there is a fundamental consistency that the trajectory must follow. Nevertheless, chaotic coupling may destroy this consistency and result in the positive conditional Lyapunov exponents. The above situation

for the Lorenz system with a double-scroll attractor also works for other coupled systems having attractors with multiple scrolls, especially for two coupled chaotic ML neurons with a multiple-scroll attractor in this paper.

With the coupling strength increasing further, two positive conditional Lyapunov exponents reach the transition values $C = 0.0630$ and 0.0885, respectively, which are characterized by the zero crossings. After the transition point at $C = 0.0630$, the second positive conditional Lyapunov exponent becomes negative as shown in Fig.3. It should be mentioned that the first conditional Lyapunov exponent is still positive until $C = 0.0885$. This implies that the random effect of chaotic coupling on synchronization still exists in a different way. From the bifurcation diagram in Fig.4(a), it is seen that in the interval $0.0630 < C < 0.0885$ there is a complicated transition process from incoherence to complete synchronization of two coupled chaotic ML neurons, and more details will be investigated in what follows.

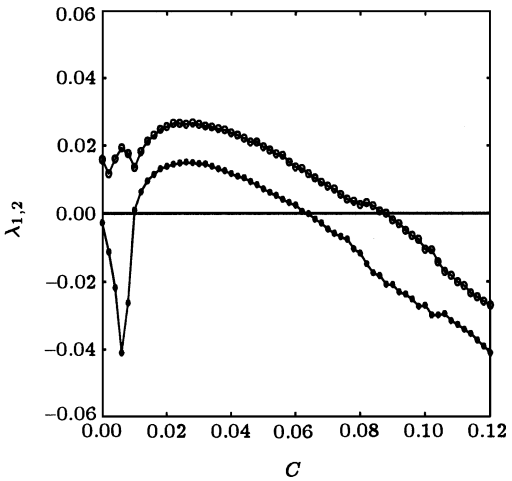


Fig.3. Two conditional Lyapunov exponents vs the coupling strength C in two coupled ML neurons when $V_{\mathrm{Ca}} = 0.845$.

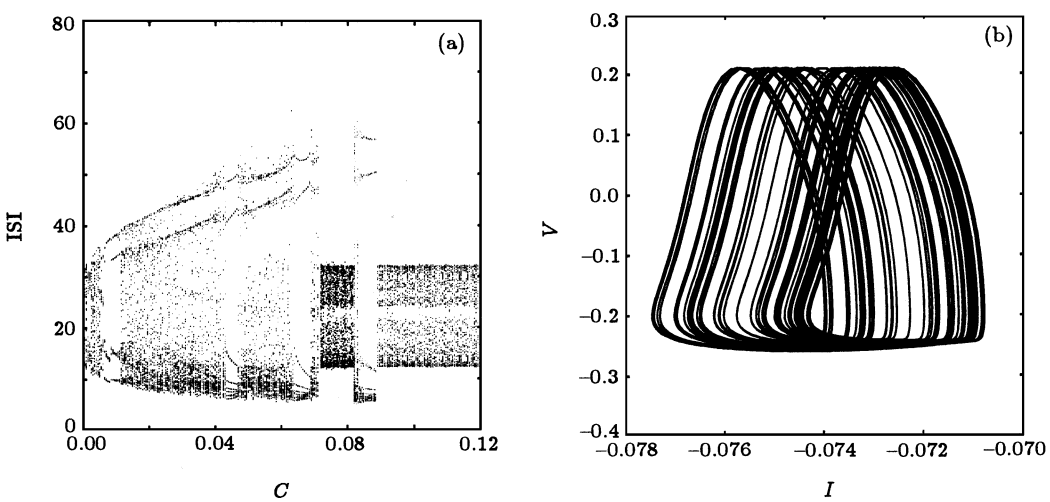


Fig.4. (a) Bifurcation diagram of ISI of the first neuron in the coupled system vs the coupling strength C. (b) The multi-scroll attractor of an ML neuron at $V_{\mathrm{Ca}} = 0.845$ for $C = 0$.

At first, it is observed that a burst synchronization appears in the interval $0.0630 < C < 0.0720$. As an example, the time series and the corresponding phase portrait are shown in Fig.5. It is seen from Fig.5(a) that the corresponding periodic bursts in two ML neurons happen simultaneously but the spikes inside may be uncorrelated. With the coupling strength C increasing further, the maximum values of the synchronization errors are nearly zero in the interval $0.0720 < C < 0.082$ as shown in Fig.6. As an example, it is seen clearly that both the trajectories on (V_1, V_2) and (I_1, I_2)-planes for $C = 0.0735$ are located in a small neighbourhood of the corresponding diagonals in Fig.7. This means the occurrence of near-synchronization in two coupled ML neurons, which are both in a chaotic state (see Fig.4(a)).

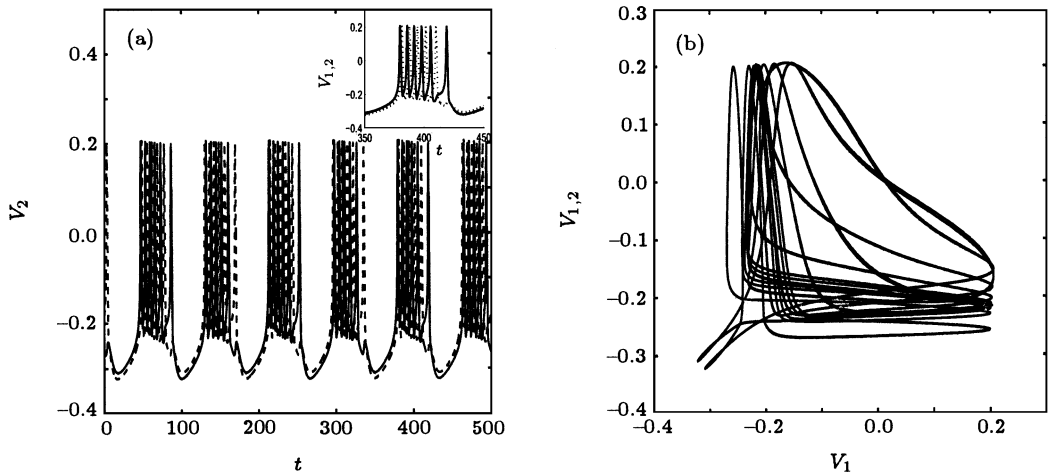


Fig.5. (a) Time series of the membrane potential of two coupled ML neurons, denoted by the solid and dashed curves, respectively, with a magnified frame in the right upper corner. (b) Phase portrait on (V_1, V_2)-plane. The coupling strength is $C = 0.0650$.

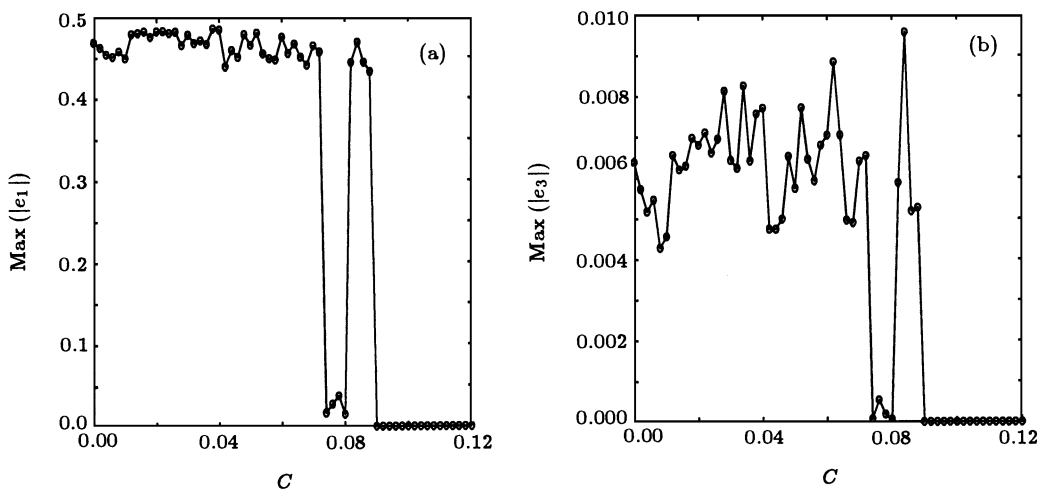


Fig.6. (a) Maximum of the synchronization error e_1 vs the coupling strength C. (b) Maximum of the synchronization error e_3 vs the coupling strength C.

When the coupling strength C goes beyond 0.082, it is seen from Fig.6 that there are sudden jumps of the synchronization errors, which is a signal of desynchronization in two coupled ML neurons. However, this desynchronization occurs only in a very small interval for $C \in (0.082\ 0.0885)$ and is embedded between near-synchronization and complete synchronization. This phenomenon may be called embedded burst synchronization of two coupled neurons as shown in Fig.4(a) and Fig.8. The near-synchronization can be considered as a transition from a burst synchronization to another one with a variation of bursting periods. This somewhat resembles the transition mechanism of period adding chaos. Hence, the zero crossing of the second conditional Lyapunov exponent can predict the onset of a transition process from incoherence to complete synchronization, which consists of burst synchronization, near-synchronization and embedded burst synchronization in two chaotic ML neurons.

When the coupling strength C is greater than 0.0885, both of the conditional Lyapunov exponents become negative with the synchronization errors vanishing (see Figs.3 and 6). This implies the realization of complete synchronization of two coupled chaotic ML neurons. The phase portraits of complete synchronization on (V_1, V_2) and (I_1, I_2)-planes at $C = 0.0920$ are shown in Fig.9.

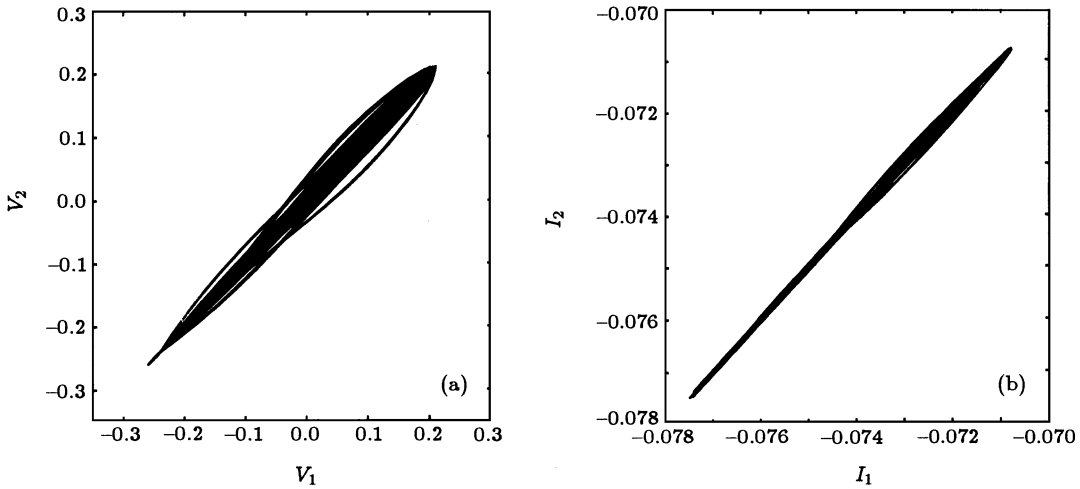


Fig.7. (a) Phase portrait on (V_1, V_2)-plane and (b) phase portrait on (I_1, I_2)-plane in two coupled ML neurons. The coupling strength is $C = 0.0735$.

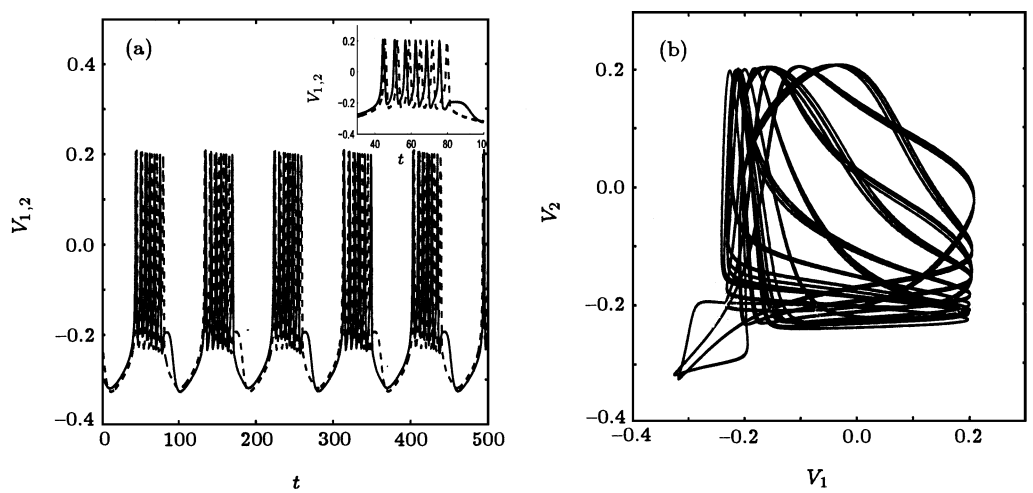


Fig.8. (a) Time series of the membrane potential of two coupled ML neurons, denoted by the solid and dashed curves, respectively, with a magnified frame in the right upper corner. (b) Phase portrait on (V_1, V_2)-plane in two coupled ML neurons. The coupling strength is $C = 0.0840$.

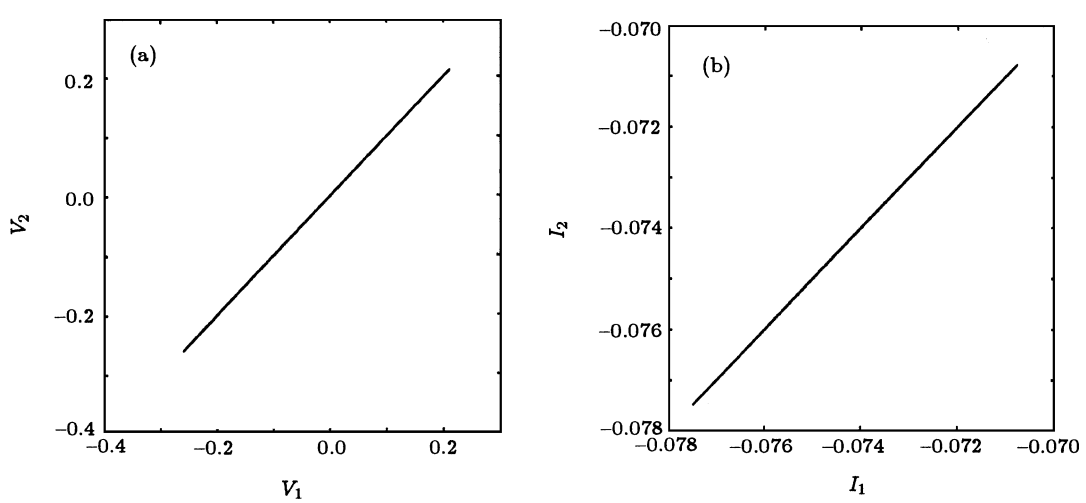


Fig.9. (a) Phase portrait on (V_1, V_2)-plane and (b) phase portrait on (I_1, I_2)-plane in two coupled ML neurons. The coupling strength is $C = 0.0920$.

A further remark is made as follows. The type of chaos synchronization in coupled multi-time scale systems is closely related to the dynamic behaviours of the fast and slow subsystems. It is reported in Ref.[10] recently that with the coupling strength increasing, there were two transition points from incoherence to complete synchronization in two coupled chaotic HR neurons, at which one of two positive Lyapunov exponents became negative. At the first transition, there was an onset of burst synchronization (namely, synchronization in the slow subsystem) but no spike synchronization (namely, synchronization in the fast subsystem). Furthermore, complete synchronization of both the slow and fast subsystems achieved directly after the second transition, without any intermediate process between burst synchronization and complete synchronization. However, the synchronization transition style of two coupled chaotic ML neurons in the present paper is somewhat different from and more complicated than that in Ref.[10]. Especially, except burst synchronization, there exists an additional intermediate process between burst synchronization and complete synchronization, which consists of near-synchronization and embedded burst synchronization in succession.

4. Conclusion

In conclusion, chaos synchronization of two coupled chaotic ML neurons is studied with the coupling strength increasing. The effect of the coupling strength on synchronization is analysed by means of the variation of the conditional Lyapunov exponents and the synchronization errors. The incoherence caused by two positive conditional Lyapunov exponents at lower coupling strengths is explained by means of the existence of a multi-scroll attractor. A new transition style from incoherence to complete synchronization is found in this system, that is, the order for realizing synchronization in two coupled chaotic ML neurons with the coupling strength increasing is observed as follows: incoherence $\longrightarrow$ burst synchronization $\longrightarrow$ near-synchronization $\longrightarrow$ embedded burst synchronization $\longrightarrow$ complete synchronization. This result reveals the mutual relation between slow and fast subsystems in transitions of chaos synchronization, and may be instructive for the study of chaos synchronization in coupled multi-time scale systems.

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